Archimedes Principle

Archimedes Principle is a very important result that follows directly from the way in which the pressure varies from point to point within a fluid at rest (see section (Cb)). We first address the simple case of an arbitrarily-shaped, impermeable body submerged in an incompressible liquid of density, ρ, which is at rest and in which the gravitational acceleration, g, is uniform so that $p = p_0 - \rho g y$ where p_0, ρ and g are constants. First we consider dividing the volume of the body into narrow vertical cylinders as shown in Figure 1. Consider one such cylinder with horizontal cross-sectional area, dA, as indicated.

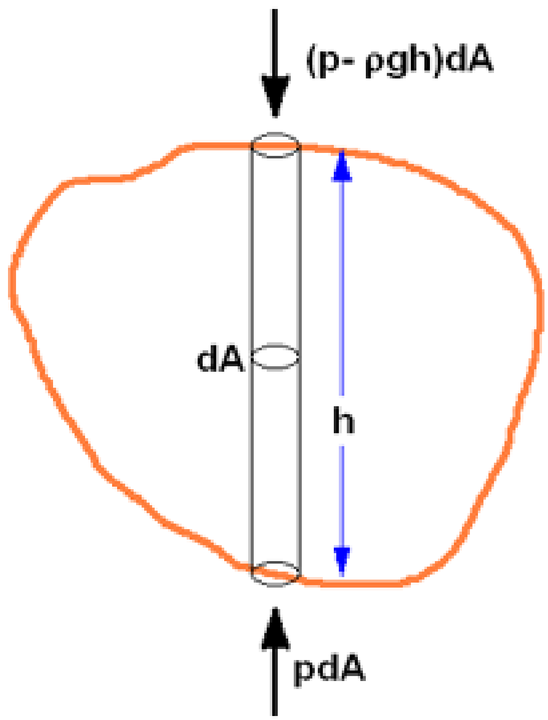

Figure 1: A submerged body divided into small, vertical, cylindrical elements, dA.

If we denote the pressure at the bottom end of this cylinder by p then the vertically upward force on that end is simply pdA whatever the shape or orientation of that end. Moreover, because of the above way in which the pressure in the fluid varies with y the pressure acting downward at the top end of the cylinder will be $(p - \rho g h)$ where h is the height of the cylinder. Consequently the force acting downward on the top will be $(p - \rho g h)dA$, again independent of the shape or orientation of that top end. Thus the net vertically-upward fluid force acting on the cylinder, dF, is

$$dF = pdA - (p - \rho g h)dA = \rho g(hdA) = \rho g dV \tag{Cg1}$$

where $hdA = dV$ is simply the volume of the cylinder. If we then sum up over all the cylinders that make up the volume of the entire submerged body, the total vertically-upward fluid force acting on the body, F, is then

$$F = \int \rho g(hdA) = \int \rho g dV = \rho g V \tag{Cg2}$$

where V is the volume of the submerged body. Moreover $\rho g V$ is simply the weight of the displaced mass of fluid and this is Archimedes principle, namely that the vertically-upward fluid force (known as the "buoyancy force") acting on a submerged body is equal in magnitude to the displaced mass of fluid.

A companion result yields the line of action of this buoyancy force. As one can confirm by considering the contribution from all the incremental cylindrical volumes to the moment acting on the body, the buoyancy force acts through the center of volume of the body (**not** the center of mass). This has important implications for the **stability of submerged bodies**.

To complete the analysis we should also consider the horizontal forces acting on the submerged body. To do this we consider subdividing the body into horizontal cylinders as shown in Figure 2. Consider any one of these cylinders with a cross-sectional area dA. If we denote the pressure at one end by p then the horizontal force acting on that end is pdA. But the other end is on the same horizontal plane and consequently the pressure there must also be p. Hence the force on the other end is also pdA and the net horizontal force is zero. Integrating over the entire volume of the submerged body the net force acting horizontally is zero as it must be.

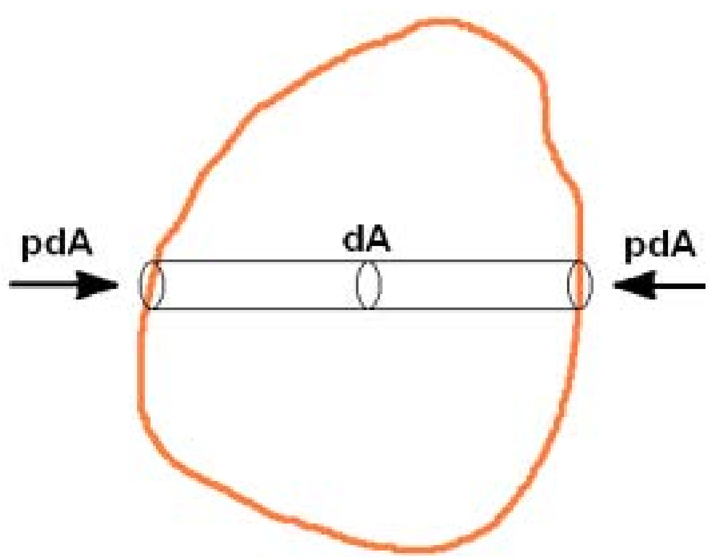

Figure 2: A submerged body divided into small, horizontal, cylindrical elements, dA.

Consider now the more general result for a fluid at rest in which the density and the acceleration due to gravity may not be constant. Again denote the pressure at the bottom of the cylinder in Figure 1 by p; the force acting on the bottom is again pdA. In this case the pressure acting on the top end of the cylinder is

$$p - \int_0^h \rho g dy \tag{Cg3}$$

where y is a coordinate measured vertically upward from the bottom end of the cylinder. Therefore it follows that the vertically upward buoyancy force on the cylindrical element is

$$dA \int_0^h \rho g dy \tag{Cg4}$$

and this is simply the sum of the weights of the displaced fluid at each point, y, along the axis of the cylinder since the incremental volume at the point y is $dAdy$, the mass of the fluid displaced locally at this point is $\rho dAdy$ and the local weight is therefore $g\rho dAdy$. Integrating over the length of the cylinder and then over all the cylinders making up the volume of the submerged body we conclude that the Archimedes principle holds whatever the spatial variations in ρ and g.

It remains to discuss the issue of **floating bodies**.