Branching Properties

The anatomy of mammals includes a multitude of branching flow networks that deliver a variety of fluids to organs and then regathers that fluid (or a modified fluid) in a return network. Examples are (1) the respiration system in the lungs that involves a blood flow network and an interacting airway network, (2) the renal network in the kidneys that cleans the blood and ejects urine through the ureter and (3) the blood supply to the muscles and other organs. Many of these networks involve a single large tube (for example the aorta or the trachea) that branches many times into a very large number to very small vessels (the *microcirculation*). In this section we explore some of the features of these branching networks.

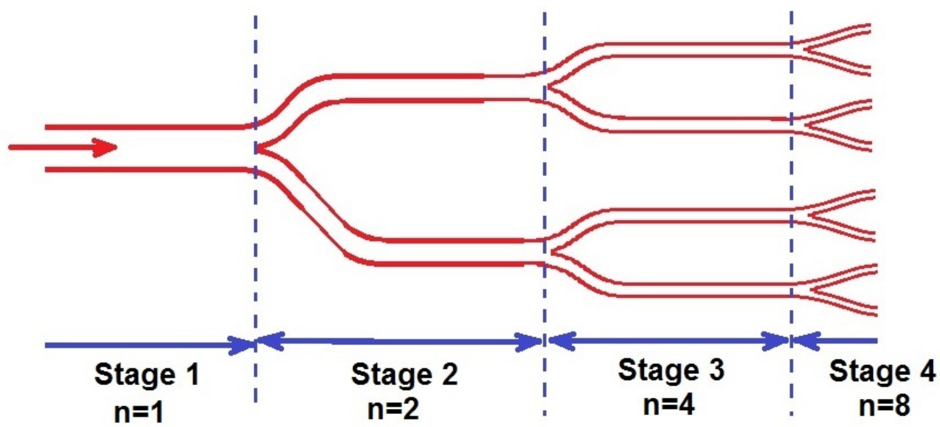

Figure 1: Branching network notation.

We model a branching network as a series of stages each of which begins and ends with a bifurcation as sketched in figure 1. We denote each stage with the index, i, the number of tubes in the stage i by n_i, the average cross-sectional area of flow in the tubes of stage i by A_i, the average radius of the tubes by R_i and the cross-sectionally averaged velocity of flow in each stage by $\overline{u}_i$. It follows that the total volume flow rate through each stage, Q, is given by

$$Q \;=\; n_i A_i \overline{u}_i \quad \text{where} \quad A_i \;=\; \pi R_i^2 \tag{Ebc1}$$

In many of these systems the flow rate, Q, through each stage is the same and the first stage ($i = 1$) involves a single tube ($n_1 = 1$) so that

$$Q \;=\; A_1 \overline{u}_1 \;=\; n_i A_i \overline{u}_i \tag{Ebc2}$$

and therefore

$$\overline{u}_i \;=\; Q/n_i A_i \tag{Ebc3}$$

The pressure gradient along the length of the tubes in stage i, $(-\partial p/\partial s)_i$ (where s is the distance along the tube axis), is conventionally given by the friction factor, f, defined by

$$f \;=\; \frac{4R_i}{\rho \overline{u}_i^2}\left(-\frac{\partial p}{\partial s}\right)_i \tag{Ebc4}$$

where ρ is the fluid density (see section (Bic)). Most of the systems considered involve laminar Poiseuille flow (see section (Bic)) for which $f = 64/Re$ and we will confine the present analysis to those flows while noting that for turbulent flow (see section (Bkl)) $f \approx 0.309/Re^{1/4}$. In these expressions the Reynolds number, Re, is defined as $Re = 2\rho R_i \overline{u}_i/\mu$ where μ is the fluid viscosity. In the laminar flow case,

$$\left(-\frac{\partial p}{\partial s}\right)_i = \frac{8\mu\overline{u}_i}{R_i^2} \quad \text{and} \quad \overline{u}_i = \frac{R_i^2}{8\mu}\left(-\frac{\partial p}{\partial s}\right)_i \tag{Ebc5}$$

It follows from equations (Ebc3) and (Ebc5) that, in each stage i, the relationship between the number of tubes, n_i, the radius of the tubes, R_i, and the pressure gradient, $(-\partial p/\partial s)_i$, is

$$n_i = \frac{8\mu Q}{\pi R_i^4 \left(-\frac{\partial p}{\partial s}\right)_i} \quad \text{or} \quad \left(-\frac{\partial p}{\partial s}\right)_i = \frac{8\mu Q}{\pi R_i^4 n_i} \tag{Ebc6}$$

The following conclusions can be drawn from equation (Ebc3) and (Ebc6). If the velocity of flow is the same at each stage then the number of tubes, n_i, would be proportional to R_i^{-2}. On the other hand if the pressure gradient were the same at each stage then the number of tubes, n_i, would be proportional to R_i^{-4}. For example, since the diameter of the aorta is $3cm$ and the diameter of the microcirculation (the smallest tubes) is $8 \times 10^{-6}m$, it would follow in the uniform velocity case that the number of tubes in the microcirculation would be 2×10^{14}. On the other hand in the uniform pressure gradient case the number of tubes in the microcirculation would be 4×10^{28}. Since most of the pressure drop in the blood circulation system occurs in the microcirculation it seems probable that the actual number of branches in the microcirculation is considerable less than the 2×10^{14} number.