Transfer Matrices

The transfer matrix for any component or device is the matrix which relates the fluctuating quantities at the discharge node to the fluctuating quantities at the inlet node. The earliest exploration of such a concept in electrical networks appears to be due to Strecker and Feldtkeller (1929) while the utilization of the idea in the context of fluid systems owes much to the pioneering work of Pipes (1940). The concept is the following. If the quantities at inlet and discharge are denoted by subscripts $i = 1$ and $i = 2$, respectively, and, if $\{\tilde{q}_i^n\}, n = 1, 2 \rightarrow N$ denotes the vector of independent fluctuating quantities at inlet and discharge for a system of order N, then the transfer matrix, $[T]$, is defined as

$$\{\tilde{q}_2^n\} = [T]\,\{\tilde{q}_1^n\} \tag{Bngc1}$$

It is a square matrix of order N. For example, for an order two system in which the independent fluctuating variables are chosen to be the total pressure, $\tilde{p}^T$, and the mass flow rate, $\tilde{m}$, then a convenient transfer matrix is

$$\begin{Bmatrix} \tilde{p}_2^T \\ \tilde{m}_2 \end{Bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{Bmatrix} \tilde{p}_1^T \\ \tilde{m}_1 \end{Bmatrix} \tag{Bngc2}$$

The words transfer function and transfer matrix are used interchangeably here to refer to the matrix $[T]$. In general it will be a function of the frequency, ω, of the perturbations and the mean flow conditions in the device.

The most convenient independent fluctuating quantities for a hydraulic system of order two are usually

1. Either the pressure, $\tilde{p}$, or the instantaneous total pressure, $\tilde{p}^T$. Note that these are related by

$$\tilde{p}^T = \tilde{p} + \frac{\bar{u}^2}{2}\tilde{\rho} + \bar{\rho}\bar{u}\tilde{u} + gz\tilde{\rho} \tag{Bngc3}$$

where $\bar{\rho}$ is the mean density, $\tilde{\rho}$ is the fluctuating density which is barotropically connected to $\tilde{p}$, and z is the vertical elevation of the system node. Neglecting the $\tilde{\rho}$ terms as is acceptable for incompressible flows

$$\tilde{p}^T = \tilde{p} + \bar{\rho}\bar{u}\tilde{u} \tag{Bngc4}$$

2. Either the velocity, $\tilde{u}$, the volume flow rate, $\bar{A}\tilde{u} + \bar{u}\tilde{A}$, or the mass flow rate, $\tilde{m} = \bar{\rho}\bar{A}\tilde{u} + \bar{\rho}\bar{u}\tilde{A} + \bar{u}\bar{A}\tilde{\rho}$. Incompressible flow at a system node in a rigid pipe implies

$$\tilde{m} = \bar{\rho}\bar{A}\tilde{u} \tag{Bngc5}$$

The most convenient choices are $\{\tilde{p}, \tilde{m}\}$ or $\{\tilde{p}^T, \tilde{m}\}$, and, for these two vectors, we will respectively use transfer matrices denoted by $[T^*]$ and $[T]$, defined as

$$\begin{Bmatrix} \tilde{p}_2 \\ \tilde{m}_2 \end{Bmatrix} = [T^*] \begin{Bmatrix} \tilde{p}_1 \\ \tilde{m}_1 \end{Bmatrix} \quad ; \quad \begin{Bmatrix} \tilde{p}_2^T \\ \tilde{m}_2 \end{Bmatrix} = [T] \begin{Bmatrix} \tilde{p}_1^T \\ \tilde{m}_1 \end{Bmatrix} \tag{Bngc6}$$

If the flow is incompressible and the cross-section at the nodes is rigid, then the $[T^*]$ and $[T]$ matrices are clearly connected by

$$T_{11} = T_{11}^* + \frac{\bar{u}_2}{A_2}T_{21}^* \quad ; \quad T_{12} = T_{12}^* - \frac{\bar{u}_1}{A_1}T_{11}^* + \frac{\bar{u}_2}{A_2}T_{22}^* - \frac{\bar{u}_1}{A_1}\frac{\bar{u}_2}{A_2}T_{21}^*$$
$$T_{21} = T_{21}^* \quad ; \quad T_{22} = T_{22}^* - \frac{\bar{u}_1}{A_1}T_{21}^* \tag{Bngc7}$$

and hence one is readily constructed from the other. Note that the determinants of the two matrices, $[T]$ and $[T^*]$, are identical.