

Unsteady Bernoulli Equation for a Pipe

To develop the equation governing the unsteady one-dimensional flow of an incompressible fluid in a duct, consider the elemental length, ds , depicted in Figure 1. Applying the linear momentum theorem to the

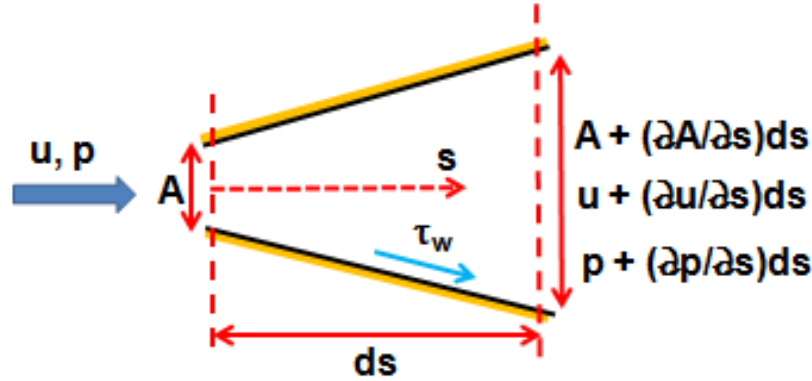


Figure 1: Arbitrary elemental length of duct.

incompressible fluid (ρ is uniform and constant) contained within the length, ds , the net force acting on the fluid in the direction, s , is

$$-A \frac{\partial p}{\partial s} ds - \tau_w P ds - \rho g A ds \frac{\partial y}{\partial s} \quad (\text{Bnda1})$$

where P is the perimeter of the cross-sectional area A , τ_w is the shear stress acting on the fluid in the direction opposite to s and y is the elevation of the duct centerline. By the linear momentum theorem this must be equal to the rate of change of momentum within the elemental volume plus the net flux of momentum out of the volume or

$$\rho A ds \frac{\partial u}{\partial t} + \rho u A ds \frac{\partial u}{\partial s} \quad (\text{Bnda2})$$

so that the governing equation becomes

$$-\frac{1}{\rho} \frac{\partial p}{\partial s} - \frac{P}{A} \frac{\tau_w}{\rho} - g \frac{\partial y}{\partial s} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial s} \quad (\text{Bnda3})$$

In addition, since the mass flow rate, $m = \rho A u$, and by conservation of mass m is independent of s , it follows that

$$\frac{\partial u}{\partial t} = \frac{1}{\rho A} \frac{dm}{dt} \quad (\text{Bnda4})$$

Also, using the definition of the friction factor, we can write,

$$\frac{\tau_w}{\rho} = \frac{1}{8} f u^2 = \frac{f}{8} \frac{m^2}{\rho^2 A^2} \quad (\text{Bnda5})$$

and assuming this adequately represents the viscous frictional effects at the duct wall for unsteady flow, the equation (Bnda3) can then be written as

$$-\frac{1}{\rho} \frac{\partial p}{\partial s} - \frac{f P}{8 A} \frac{m^2}{\rho^2 A^2} - g \frac{\partial y}{\partial s} = \frac{1}{\rho A} \frac{dm}{dt} + u \frac{\partial u}{\partial s} \quad (\text{Bnda6})$$

Since the mass flow rate, m , must be independent of s , equation (Bnda6) can then be integrated over the length of a duct of varying cross-sectional area, $A(s)$ (and perimeter $P(s)$), from the inlet (point A) to the end (point B) to yield

$$\left\{ p + \rho g y + \frac{1}{2} \rho u^2 \right\}_A - \left\{ p + \rho g y + \frac{1}{2} \rho u^2 \right\}_B = \frac{dm}{dt} \int_A^B \frac{ds}{A} + \frac{m^2}{\rho} \int_A^B \frac{fP}{8A^3} ds \quad (\text{Bnda7})$$

or

$$p_A^T - p_B^T = \frac{dm}{dt} \int_A^B \frac{ds}{A} + m^2 \int_A^B \frac{fP}{8\rho A^3} ds \quad (\text{Bnda8})$$

The second term on the right hand side is the same as occurs in steady flow as a result of the friction factor, f . The first term on the right hand side is the new term that results from the acceleration of the flow (obviously it is zero in a steady flow). In words the first term represents the work done in accelerating the flow while the second is the loss of kinetic energy due to viscous effects at the walls. The coefficient of the first term is called the **inertance** and is denoted by L and, for reasons that will become evident later, we also define a quantity called the **resistance**, R , given by:

$$L = \int_A^B \frac{ds}{A} \quad \text{and} \quad R = \frac{m}{4\rho} \int_A^B \frac{fP}{A^3} ds \quad (\text{Bnda9})$$

For example, for a straight uniform, cylindrical pipe of length, ℓ :

$$L = \frac{\ell}{A} \quad \text{and} \quad R = \frac{m}{4\rho} \frac{fP\ell}{A^3} = \frac{fuP\ell}{4A^2} \quad (\text{Bnda10})$$

and for a number of such pipes (each denoted by a subscript, i) connected in series:

$$L = \sum_i \frac{\ell_i}{A_i} \quad \text{and} \quad R = \frac{m}{4\rho} \sum_i \frac{f_i P_i \ell_i}{A_i^3} \quad (\text{Bnda11})$$

For a flow with small amplitude oscillations of frequency, ω , superimposed on a mean flow with mass flow rate, \bar{m} , the oscillations are subject to an **impedance** given by $(R + j\omega L)$ where $m = \bar{m}$ is used to evaluate R . Much more on these so-called **lumped parameter models** can be found in the section on frequency domain methods.

Equation (Bnda8) can be called the modified Bernoulli equation for unsteady, one-dimensional, incompressible flow. Perhaps a useful example of the application of this equation to an unsteady flow is the analysis of manometer oscillations.