Turbulent Boundary Layers

We now turn to describe the characteristics and analyses of turbulent boundary layers, focusing on the boundary layers on flat surfaces though the results are frequently applied to surfaces with different geometry. Figure 1 presents experimental measurements of the Reynolds stress quantities in a turbulent boundary layer as a function of distance, y, from the surface as well as the mean velocity profile, $\overline{u}/U$. The insert shows how all the Reynolds stresses begin at zero at the surface, $y = 0$, while the main graph shows how they peak a short distance from the surface and eventually decline to zero at the edge of the boundary layer. None of these fluctuation quantities are large but the Reynolds shear stress, $\overline{u'v'}/U^2$, dominates the shear stress in many turbulent flows including turbulent boundary layers.

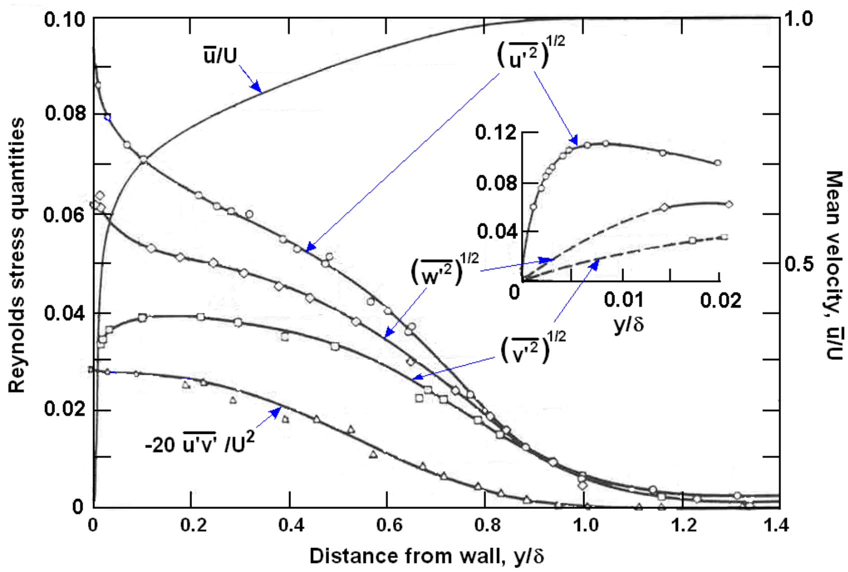

Figure 1: Measurements of the Reynolds stresses in a turbulent boundary layer.

Turbulent boundary layers, while not as simple as turbulent pipe flow since they are evolving with distance, can be treated in essentially the same way by utilizing the universal velocity profile (equations (Bkl7)

$$u^* = y^* \tag{Bkk1}$$

for $y^* < 5$ and

$$u^* = \frac{\overline{u}}{u_\tau} = 5.75 \; log_{10}(y^*) + 5.5 \tag{Bkk2}$$

for $y^* > 5$ along with the Karman momentum integral equation (equation (Bjh12)) for boundary layers. With zero pressure gradient ($\partial U/\partial s = 0$) the Karman momentum integral equation becomes

$$\frac{\tau_W}{\rho} = U^2 \frac{d\delta_M}{ds} \tag{Bkk3}$$

Defining the outer limit of the turbulent boundary layer where $\overline{u} \rightarrow U$ to be $y = \delta$ it follows from equation (Bkk2) that

$$\frac{U}{u_\tau} = 5.75 \; log_{10} \left\{ \frac{\delta}{\nu} \left(\frac{\tau_W}{\rho} \right)^{\frac{1}{2}} \right\} + 5.5 \tag{Bkk4}$$

and therefore

$$\frac{\overline{u}}{U} = \frac{log_{10}\left(yu_\tau/\nu\right) + 0.956}{log_{10}\left(\delta u_\tau/\nu\right) + 0.956} \tag{Bkk5}$$

This is the expression for $\overline{u}/U$ as a function of y that is needed with the Karman momentum integral equation to solve for the flow. Unfortunately, it does not necessarily conform to a simple and constant numerical relation between δ and δ_M that would permit progress toward a solution. Even if we assume such a relation, namely that $\delta_M = \alpha\delta$ where α is an assumed, known constant, the differential equation for $\delta(s)$ that results is

$$\left(\alpha\frac{d\delta}{ds}\right)^{-\frac{1}{2}} = 5.75\ log_{10}\left\{\left(\frac{\delta U}{\nu}\right)\left(\alpha\frac{d\delta}{ds}\right)^{\frac{1}{2}}\right\} + 5.5 \tag{Bkk6}$$

which is not soluble analytically.

However, if we take a simpler approach and assume that that the velocity profiles in the boundary layer can be approximated by the Blasius one-seventh power law profile, equation (Bkj9)

$$u^* = 8.7(y^*)^{\frac{1}{7}} \qquad \text{or} \qquad \frac{\overline{u}}{u_\tau} = 8.7\left(\frac{yu_\tau}{\nu}\right)^{\frac{1}{7}} \tag{Bkk7}$$

and therefore

$$\frac{U}{u_\tau} = 8.7\left(\frac{\delta u_\tau}{\nu}\right)^{\frac{1}{7}} \tag{Bkk8}$$

so that

$$\frac{\overline{u}}{U} = 8.7\left(\frac{y}{\delta}\right)^{\frac{1}{7}} \tag{Bkk9}$$

Consequently the Blasius profile leads to self-similar velocity profiles and therefore to a simple, constant value of $\alpha = 7/72 = 0.0972$. Moreover from the relation (Bkk8) it follows that

$$\frac{\tau_W}{\rho U^2} = 0.0228\left(\frac{\delta U}{\nu}\right)^{-\frac{1}{4}} \tag{Bkk10}$$

and with the Karman momentum integral equation (Bkk3) this leads to the following differential equation for the boundary layer thickness, δ:

$$0.0972\frac{d\delta}{ds} = 0.0228\left(\frac{\delta U}{\nu}\right)^{-\frac{1}{4}} \tag{Bkk11}$$

whose solution is

$$\delta^{\frac{5}{4}} = 0.294\left(\frac{\nu}{U}\right)^{\frac{1}{4}} s + C \tag{Bkk12}$$

where C is an integration constant to be determined at the location $s = s_0$ where the layer first becomes turbulent. If $\delta = \delta_0$ at $s = s_0$ then

$$\delta^{\frac{5}{4}} - \delta_0^{\frac{5}{4}} = 0.294\left(\frac{\nu}{U}\right)^{\frac{1}{4}}(s - s_0) \tag{Bkk13}$$

Often, at high Reynolds numbers, both δ_0 and s_0 are negligibly small so that

$$\delta^{\frac{5}{4}} \approx 0.294\left(\frac{\nu}{U}\right)^{\frac{1}{4}} s \tag{Bkk14}$$

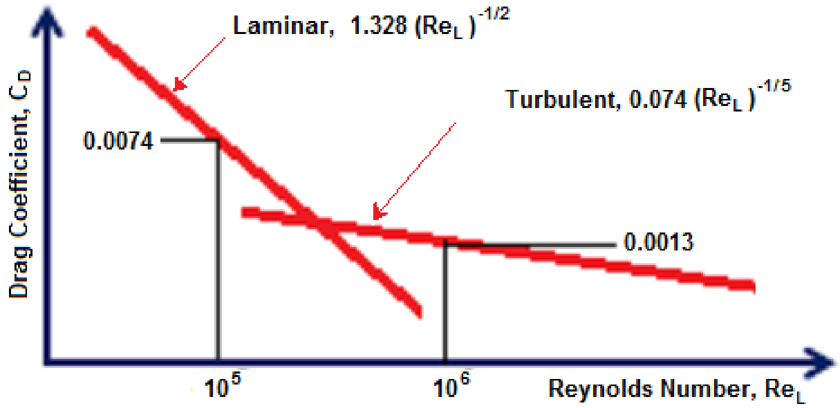

Figure 2: Plot of the drag coefficient due to skin friction on a flat plate as a function of the Reynolds number, $Re_L = UL/\nu$.

Among the other useful results of this turbulent boundary layer analysis is the surface shear stress which, from the relations (Bkk10) and (Bkk14), becomes:

$$\frac{\tau_W}{\frac{1}{2}\rho U^2} \;=\; 0.058\left(\frac{\nu}{Us}\right)^{\frac{1}{5}} \;=\; 0.058(Re_s)^{-\frac{1}{5}} \qquad \text{(Bkk15)}$$

where Re_s is the Reynolds number based on the distance along the surface. The left-hand side defines what is known as the skin-friction coefficient. It should be compared with the skin-friction coefficient for laminar boundary layer flow, namely $0.664(Re_s)^{-\frac{1}{2}}$. This comparison shows that the surface friction decreases less quickly with distance, s, under the turbulent boundary layer and this helps explain why separation (which occurs when τ_W declines to zero) is delayed in a turbulent boundary layer in comparison to a laminar boundary layer.

The skin friction drag, D, on a plate of length L and breadth B due to the turbulent boundary layer on one side of the plate (assuming both δ_0 and s_0 are zero) is consequently given by

$$D \;=\; B\int_0^L \tau_W \; ds \qquad \text{(Bkk16)}$$

and substituting the expression (Bkk15) the drag coefficient, $C_D = 2D/\rho LBU^2$, is given by

$$C_D \;=\; 0.074(Re_L)^{-\frac{1}{5}} \qquad \text{(Bkk17)}$$

where Re_L is the Reynolds number based on the length of the plate, L. This compares with the equivalent expression for a laminar boundary layer namely $C_D = 1.325(Re_L)^{-\frac{1}{2}}$. Figure 2 presents how these results appear in a graph of the drag coefficient against the Reynolds number.