Planar Stokes Flow

As we observed in section (Bla), the basic governing equation for incompressible, planar Stokes flow is the biharmonic equation for the stream function:

$$\nabla^2 \left\{ \nabla^2 \psi \right\} \; = \; 0 \tag{Blf1}$$

Thus any incompressible, planar *potential* flow (section (Bga)) solution to $\nabla^2\psi = 0$ is also a solution to some Stokes flow. However, in most flows of practical interest, there is a problem with satisfying the boundary conditions. Both Stokes' flow and incompressible, planar *potential* flow must satify the condition of zero normal velocity at a solid surface. But Stokes flows must also satisfy the no-slip condition and the aforementioned incompressible, planar *potential* flows do not satisfy that condition. Therefore, although a potential flow satisfying $\nabla^2\psi = 0$ will always satisfy the governing field equation (Blf1) for Stokes' flow, that potential flow solution will not generally satisfy the boundary conditions required of a Stokes' flow solution.

However, it is appropriate to point out that some of the techniques of complex variables that were described in section (Bgeb) and used to good effect in the context of incompressible, planar *potential* flow may be useful in planar Stokes' flow. It is readily shown that if we define a complex position vector, $z = x + iy$, in the xy plane of a planar Stokes flow then the general solution of the governing biharmonic equation can be written as:

$$\psi(x, y) \; = \; Re \left\{ \overline{z} f(z) + g(z) \right\} \tag{Blf2}$$

where $f(z)$ and $g(z)$ may be any analytic functions (note that the imaginary part is also a solution). However, the utility of this is much less than the equivalent result in planar potential flow, $\psi = Re\{f(z)\}$.

We could also seek to find solutions by separation of variables as was done for planar potential flow in section (Bgda). Thus we should seek a solution of the form

$$\psi \; = \; R(r,t) \; \Theta(\theta, t) \tag{Blf3}$$

where r and θ are polar coordinates in the xy plane so that $x = r\cos\theta$ and $y = r\sin\theta$ and the functions $R(r,t)$ and $\Theta(\theta,t)$ are to be determined. Substituting into the biharmonic equation

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right] \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right] \psi \; = \; 0 \tag{Blf4}$$

leads to

$$\frac{1}{R}\left\{ r^4 \frac{d^4R}{dr^4} + 2r^3\frac{d^3R}{dr^3} - r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} \right\} + \frac{1}{R}\left\{ 4R + 2r\frac{dR}{dr} + r^2\frac{d^2R}{dr^2} \right\} \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} + \frac{1}{\Theta}\frac{d^4\Theta}{d\theta^4} \; = \; 0 \tag{Blf5}$$

which has single-valued solutions (in the sense that they involve sines and cosines of integers of θ) of the form

$$\psi \; = \; C_{10} + C_{20}r^2 + C_{30}\ln r + \sum_k (C_{1k}r^k + C_{2k}r^{-k} + C_{3k}r^{2+k} + C_{4k}r^{2-k})(\sin k\theta + C_{5k}\cos k\theta) \tag{Blf6}$$

where the k quantities are integers and all the C_{xx} quantities are constants to be determined. Then, if we choose to find the flow around a cylinder of radius R in a uniform stream of velocity, U, it can be seen that there is no solution which satisfies the conditions of a uniform stream velocity as $r \rightarrow \infty$ and the conditions of $u_r = (1/r)\partial\psi/\partial\theta = 0$ and $u_\theta = -\partial\psi/\partial r = 0$ (the zero normal and no-slip velocities) at the cylinder surface. This simply reflects the *Whitehead paradox* that there is no solution for the Stokes flow of a uniform stream around a cylinder. However, there may be other Stokes flows for which the solution of equation (Blf6) is useful.