Equations of Stokes Flow

The equations governing Stokes flow are a continuity equation

$$\frac{\partial u_i}{\partial x_i} = 0 \qquad \text{or} \qquad \nabla.\underline{u} = 0 \tag{Bla1}$$

and an equation of motion

$$-\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} = 0 \qquad \text{or} \qquad -\nabla p + \mu \nabla^2 \underline{u} = 0 \tag{Bla2}$$

where we have dropped the term due to a conservative force field like gravity since it is readily incorporated into the pressure.

For a planar flow in the xy plane the continuity equation becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{Bla3}$$

and therefore, as in all incompressible planar flows, we can define a streamfunction, $\psi(x, y)$, such that

$$u = \frac{\partial \psi}{\partial y} \qquad ; \qquad v = -\frac{\partial \psi}{\partial x} \tag{Bla4}$$

and the equations of motion become

$$-\frac{\partial p}{\partial x} + \mu \nabla^2 \left\{ \frac{\partial \psi}{\partial y} \right\} = 0 \qquad \text{and} \qquad -\frac{\partial p}{\partial y} - \mu \nabla^2 \left\{ \frac{\partial \psi}{\partial x} \right\} = 0 \tag{Bla5}$$

where the operator, ∇^2, denotes $\partial^2/\partial x^2 + \partial^2/\partial y^2$ and we have dropped the external force field terms, f_i, by assuming that the force field is conservative so that the effects of those terms can be absorbed into the pressure terms. Moreover the pressure can be eliminated from equations (Bla5) to yield a basic governing equation for incompressible, planar Stokes flow:

$$\nabla^2 \left\{ \nabla^2 \psi \right\} = 0 \tag{Bla6}$$

Recalling that the corresponding governing equation for incompressible, planar *potential* flow (section (Bga)) is $\nabla^2 \psi = 0$, Stokes flows governed by equation (Bla6) require *two* boundary conditions on all solid boundaries (of known location) rather than the single boundary condition required for the potential flow solution. Specifically, Stokes flows require not only the condition of zero velocity normal to and relative to that solid surface but also the *no slip condition* or zero velocity tangential to and relative to that solid surface. Therefore, although a potential flow satisfying $\nabla^2 \psi = 0$ will always satisfy the governing field equation (Bla6) for Stokes flow, that potential flow solution will not necessarily satisfy the boundary conditions required of a Stokes' flow solution.

A couple of additional properties of Stokes' flows are worth noting before proceeding. First we note that since the inertial terms have been neglected, the flow has no "memory" and is therefore in equilibrium at all times. Therefore the solution to an unsteady Stokes flow is simply a sequence of steady flow solutions that satisfy the boundary conditions at each moment in time. For the same reason, Stokes flows are reversible in time and this can be a useful tool in the analysis of some Stokes' flows.