Point Singularities

To this point we have focussed entirely on the simulation of planar potential flows. However, the methodology is readily extended to three-dimensional potential flows. We begin with a point source (or sink) of strength, Q, which is the volume of incompressible fluid leaving the source per unit time. In the absence of any other disturbances this produces a spherically-symmetric flow with velocity, $u_r = Q/4\pi r^2$, which therefore has a velocity potential given by

$$\phi \;=\; -\frac{Q}{4\pi r} \;=\; -\frac{Q}{4\pi(x^2+y^2+z^2)^{\frac{1}{2}}} \qquad \text{(Bgdn1)}$$

If this point source is placed in a uniform stream with velocity, U, in the positive x direction, the combined velocity potential of the resulting axisymmetric flow is

$$\phi \;=\; Ux - \frac{Q}{4\pi(x^2+y^2+z^2)^{\frac{1}{2}}} \qquad \text{(Bgdn2)}$$

which has velocity components

$$u \;=\; U + \frac{Qx}{4\pi(x^2+y^2+z^2)^{\frac{3}{2}}} \quad ; \quad v \;=\; \frac{Qy}{4\pi(x^2+y^2+z^2)^{\frac{3}{2}}} \quad ; \quad w \;=\; \frac{Qz}{4\pi(x^2+y^2+z^2)^{\frac{3}{2}}} \qquad \text{(Bgdn3)}$$

The typical streamlines for this flow are shown in Figure 1 which is very similar to the planar flow generated by a line source in a uniform stream. Moreover, the streamline dividing the region in which

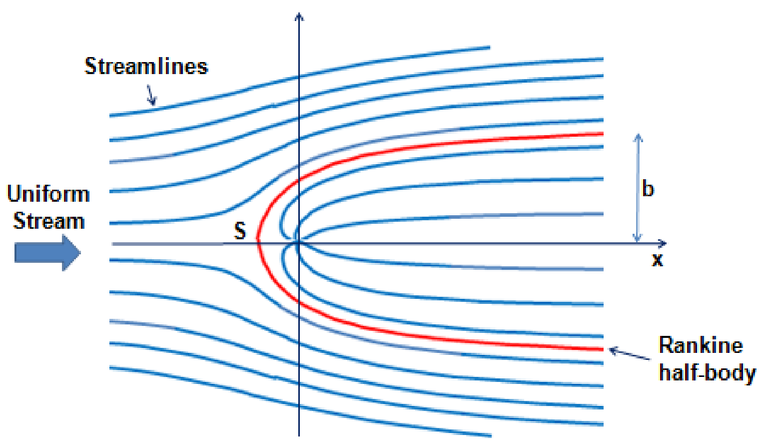

Figure 1: Axisymmetric potential flow of a source (at the origin) in a uniform stream showing streamlines and the Rankine half-body in red.

the flow originates upstream from that in which the flow originates in the source defines an axisymmetric Rankine half-body as shown in red in Figure 1.

Continuing the development of the axisymmetric flows that parallel the earlier planar flows, it follows that a source at $x = -a$, $y = z = 0$ plus a sink at $x = +a$, $y = z = 0$ has a velocity potential given by

$$\phi \;=\; \frac{Q}{4\pi}\left[\left\{(x-a)^2+y^2+z^2\right\}^{-\frac{1}{2}} - \left\{(x+a)^2+y^2+z^2\right\}^{-\frac{1}{2}}\right] \qquad \text{(Bgdn4)}$$

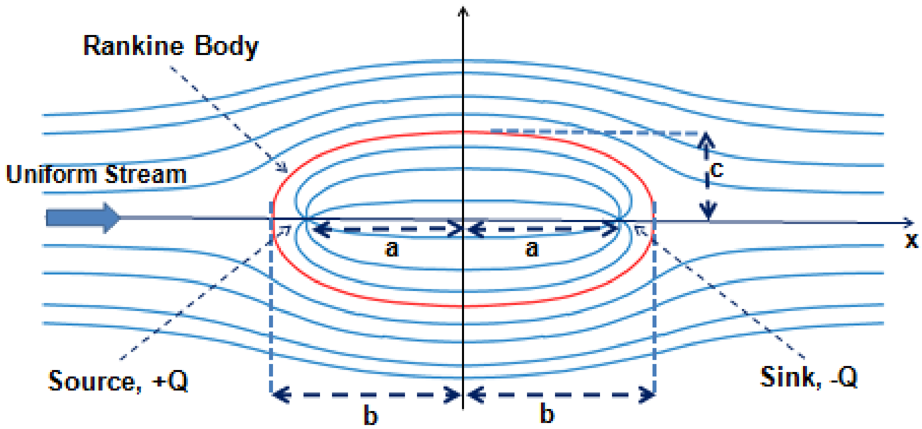

Figure 2: Axisymmetric potential flow of a source and sink of equal strength in a uniform stream showing streamlines and the Rankine body in red.

and, adding a uniform stream with velocity potential Ux, produces streamlines such as those of Figure 2. This yields the finite, axisymmetric Rankine body (or Rankine "ovoid") shown in red.

Returning to equation (Bgdn2) and the combination of a source and a sink, if we let $a \to 0$ while maintaining a finite value of Qa we produce a point doublet oriented in the x direction with a velocity potential

$$\phi = \frac{Qa}{2\pi}\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}} \qquad \text{(Bgdn5)}$$

Note that, as in the case of line sources, sinks and doublets, an x-oriented doublet is simply the differentiation of a source in the x direction. As with line singularities, an infinite array of increasingly weak singularities results for continued differentiation. Adding a uniform stream potential, Ux, to this we obtain the following potential

$$\phi = Ux + \frac{Qa}{2\pi}\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}} = Ur\cos\theta + \frac{Qa}{2\pi}\frac{\cos\theta}{r^2} \qquad \text{(Bgdn6)}$$

where (r,θ) are polar coordinates based at the origin $x=y=z=0$ with θ being the angle between the radial vector and the x axis. The velocities associated with this velocity potential are

$$u_r = \frac{\partial\phi}{\partial r} = U\cos\theta - \frac{Qa}{\pi}\frac{\cos\theta}{r^3} \qquad \text{(Bgdn7)}$$

$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\sin\theta - \frac{Qa}{2\pi}\frac{\sin\theta}{r^3} \qquad \text{(Bgdn8)}$$

In a manner similar to the planar flow case, we note that u_r is zero at a particular radius, in this axisymmetric case at $r = (Qa/\pi U)^{\frac{1}{3}}$ and thus we have obtained the solution for potential flow around a sphere. Denoting the radius of the sphere by $R = (Qa/\pi U)^{\frac{1}{3}}$ the potential flow solution which is sketched in Figure 3 becomes

$$\phi = U\cos\theta\left[r + \frac{R^3}{2r^2}\right] \qquad \text{(Bgdn9)}$$

$$u_r = U\cos\theta\left[1 - \frac{R^3}{r^3}\right] \qquad \text{(Bgdn10)}$$

$$u_\theta = -U\sin\theta\left[1 + \frac{R^3}{2r^3}\right] \qquad \text{(Bgdn11)}$$

Note that the magnitude of the velocity on the surface of the sphere is $\frac{3}{2}\sin\theta$ and therefore the coefficient of pressure on the surface is

$$(C_p)_{r=R} \;=\; \frac{(p)_{r=R} - p_\infty}{\frac{1}{2}\rho U^2} \;=\; 1 - \frac{9}{4}\sin^2\theta \tag{Bgdn12}$$

This and other flows around a sphere are discussed in detail in other sections of the book.

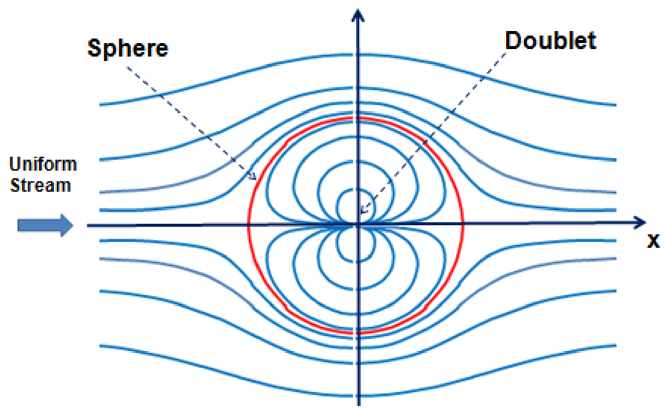

Figure 3: Axisymmetric potential flow past a sphere.

In conclusion we note that many "airship" shapes can be generated by distributions of sources and sinks along an axis. Indeed, in order to ensure a finite body in which the total source strength is balanced by the total sink strength, it is marginally better to use a distribution of doublets for which that condition is automatically satisfied. These methods were in fact used in the early days to design actual airships and to determine the forces on the structure of dirigibles. Later, as greater computational tools became available in the second half of the 20th century, more complex shapes were produced in efforts to simulate the three-dimensional flows around aircraft.