General Solution to Potential Flow in Polar Coordinates

Separation of variables in polar coordinates allowed us to construct the following solution for planar potential flows:

$$\phi = (C_{3k} r^k + C_{4k} r^{-k})(C_{1k} \sin k\theta + C_{2k} \cos k\theta) \tag{Bgdb1}$$

where the velocities in the r and θ directions, u_r and u_θ, are

$$u_r = \frac{\partial \phi}{\partial r} = k(C_{3k} r^{k-1} - C_{4k} r^{-(k+1)})(C_{1k} \sin k\theta + C_{2k} \cos k\theta) \tag{Bgdb2}$$

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = k(C_{3k} r^{k-1} + C_{4k} r^{-(k+1)})(C_{1k} \cos k\theta - C_{2k} \sin k\theta) \tag{Bgdb3}$$

and the quantities C_{1k}, C_{2k}, C_{3k}, and C_{4k}, may be constants or functions of time. To complete the documentation we can construct the streamfunction, ψ, for this planar, incompressible flow. Recall that

$$u_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad \text{and} \quad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} \tag{Bgdb4}$$

and therefore by integrating the expressions (Ida7) and (Ida8) we can obtain

$$\psi = (C_{3k} r^k + C_{4k} r^{-k})(-C_{1k} \cos k\theta + C_{2k} \sin k\theta) \tag{Bgdb5}$$

The additional subscript k has been added to all the C constants (or functions of time) since we will be superimposing solutions with different k values and these do not necessarily have the same C constants.

In addition there is the special case of $k = 0$ which yields the solution

$$\phi = (C_{30} + C_{40} \ln r)(C_{10} + C_{20}\theta) \tag{Bgdb6}$$

with velocities

$$u_r = \frac{\partial \phi}{\partial r} = \frac{C_{40}}{r}(C_{10} + C_{20}\theta) \tag{Bgdb7}$$

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = (C_{30} + C_{40} \ln r)\frac{C_{20}}{r} \tag{Bgdb8}$$

$$\psi = -C_{20}(C_{30} \ln r + \frac{1}{2} C_{40} (\ln r)^2) + C_{40}(C_{10}\theta + \frac{1}{2} C_{20}\theta^2) \tag{Bgdb9}$$

Indeed, by superposition we could write a more complex potential flow solution of this kind as

$$\phi = (C_{30} + C_{40} \ln r)(C_{10} + C_{20}\theta) + \sum_{k=1}^{K} (C_{3k} r^k + C_{4k} r^{-k})(C_{1k} \sin k\theta + C_{2k} \cos k\theta) \tag{Bgdb10}$$

$$u_r = \frac{\partial \phi}{\partial r} = \frac{C_{40}}{r}(C_{10} + C_{20}\theta) + \sum_{k=1}^{K} k(C_{3k} r^{k-1} - C_{4k} r^{-(k+1)})(C_{1k} \sin k\theta + C_{2k} \cos k\theta) \tag{Bgdb11}$$

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = (C_{30} + C_{40} \ln r)\frac{C_{20}}{r} + \sum_{k=1}^{K} k(C_{3k} r^{k-1} + C_{4k} r^{-(k+1)})(C_{1k} \cos k\theta - C_{2k} \sin k\theta) \tag{Bgdb12}$$

$$\psi \;=\; -C_{20}(C_{30}\ln r + \frac{1}{2}C_{40}(\ln r)^2) + C_{40}(C_{10}\theta + \frac{1}{2}C_{20}\theta^2) + \sum_{k=1}^{K}(C_{3k}r^k + C_{4k}r^{-k})(-C_{1k}\cos k\theta + C_{2k}\sin k\theta) \tag{Bgdb13}$$

But we could then go further and, again by superposition, add together expressions such as equations (Bgdb10), (Bgdb11), (Bgdb12) and (Bgdb13) for polar coordinates with origins in many different places in the plane of the flow. For present purposes, however, equations (Bgdb10), (Bgdb11), (Bgdb12) and (Bgdb13) will be sufficient to demonstrate examples of the flows that can be generated by this means.

Several footnotes are appropriate before proceeding to examples of these potential flows:

1. First note that some of the terms involving θ in equations (Bgdb10), (Bgdb11), (Bgdb12) and (Bgdb13) imply that the velocities have multiple values since they would give a different velocity if θ is increased by 2π. Thus we usually need to eliminate that possibility by setting $C_{20} = 0$. For the same reason we must confine the summation to integer values of k so that the velocities are the same when θ is increased by 2π.

2. Note also that these solutions become singular at either the origin, $r = 0$, or at infinity, $r \rightarrow \infty$. Since infinite velocities would be inappropriate, that location (the origin or infinity) must lie outside the domain of the flow. The first step in accomplishing this is to stipulate whether we wish to construct an **external** flow or an **internal** flow:

   - An **external** flow is one in which the flow domain is outside of a body or object and the origin of the coordinate system(s), $r = 0$, used to generate the solution using equations (Bgdb10), (Bgdb11), (Bgdb12) and (Bgdb13) lies inside the body and therefore outside of the domain of the flow. Consequently the constants, C_{4k} and C_{40}, can be non-zero without generating unrealistic infinite velocities. However, if the flow extends all the way to infinity, then the constants, C_{3k}, must be set equal to zero in order in avoid the solution blowing up as $r \rightarrow \infty$.
   - An **internal** flow is the obverse of the above. It is one in which the flow domain is inside of a containing vessel so that $r \rightarrow \infty$ outside of the domain of the flow. Consequently the constants, C_{3k}, can be non-zero without generating unrealistic infinite velocities. However, if the origin $r = 0$ is inside of the domain of the flow then the constants, C_{4k} and C_{40}, must be set equal to zero in order in avoid the solution blowing up at that origin.
   - Because of the aforementioned singular behavior at either the origin or at infinity, the construction of a potential flow solution using elements of equations (Bgdb10), (Bgdb11), (Bgdb12) and (Bgdb13) is often called the method of singularities and the individual elements which blow up at the origin (or infinity) are referred to as singularities. We explore the characteristics of many of these singularities in the sections which follow.

3. As previously discussed, potential flow only requires the application at a solid boundary of the condition of zero normal velocity. Since that condition is automatically satisfied on any streamline it follows that, in potential flow, we can replace any streamline (or streamlines) by a solid surface and thereby construct the flow which has that streamline as a solid boundary. In that way many flow solutions with solid boundaries can be generated. Note that this is not necessarily true for solutions of a viscous flow which must also satisfy the no-slip condition at a solid boundary.