Conformal Mapping

By utilizing the process of conformal mapping, we can substantially enhance our ability to find solutions to planar potential flows by the method of complex variables. In its simplest form the process involves taking a known potential flow solution in, say, the $z = x + iy$ plane and conformally mapping that geometry into a different geometry in, say, the $\zeta = \xi + i\eta$ plane with a known conformal mapping function, $\zeta(z)$. One of the characteristics of a conformal mapping is that all the angles are preserved as illustrated in Figure 1.

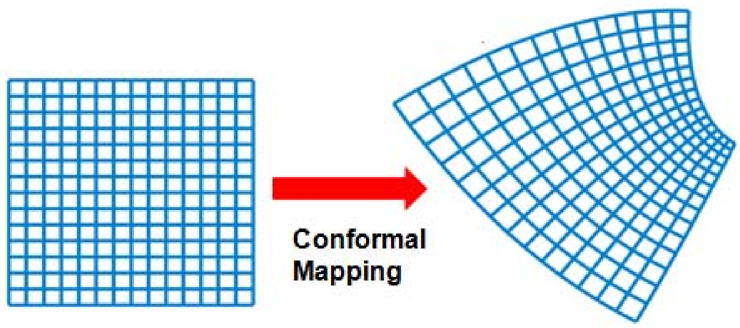

Figure 1: Illustration of conformal mapping.

It follows that since $f(z)$ is a solution to potential flow, so too is $f(\zeta)$ and, if the geometry in the ζ plane represents the desired flow geometry, we have obtained the desired potential flow solution. In some cases, it may not be easy or even possible to find the mapping $\zeta(z)$ which produces the desired geometry and that represents the principal challenge of these analytical methods. However, there are many useful solutions which can be accessed using conformal mapping and we will illustrate this by some examples.

One of the simplest examples is to begin with a uniform stream, $f(z) = Uz$, bounded by a straight wall

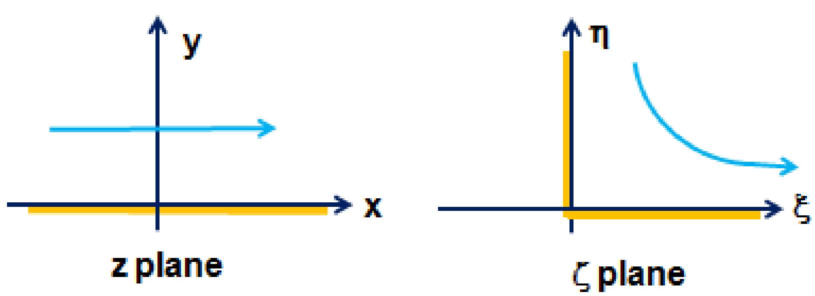

Figure 2: Mapping a uniform stream to the flow in a corner.

along the x-axis. If we deploy the mapping $\zeta = z^{\frac{1}{2}}$, this maps the part of the wall on the positive x axis onto the positive ξ axis in the ζ plane and the part of the wall on the negative x axis onto the positive η axis as shown in Figure 2. Therefore we have generated the potential flow in a right-angle corner, namely $f(\zeta) = U\zeta^2$ which is clearly the solution we discussed much earlier for which $\phi = U(\xi^2 - \eta^2)$ and $\psi = 2U\xi\eta$. Moreover, it is readily seen that a more general mapping like $\zeta = z^{\frac{1}{n}}$ will produce the flows in wedge-shaped regions discussed in the preceding sections.

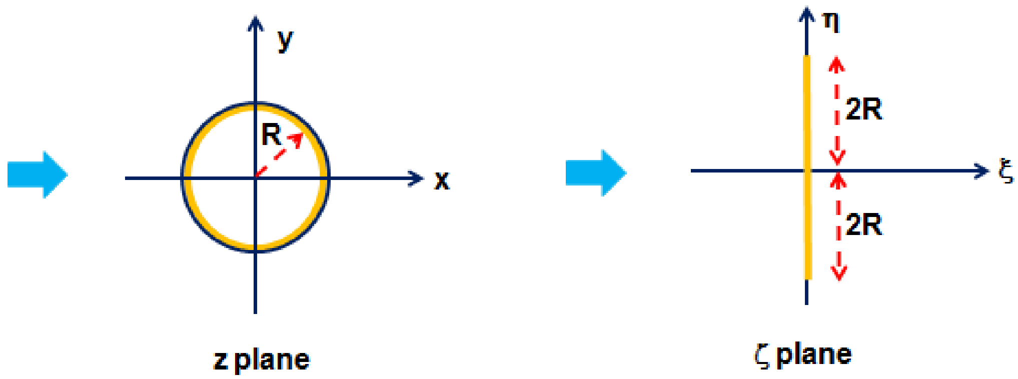

Figure 3: Mapping flow around a cylinder to flow past a flat plate.

As another example, we begin with the previously derived potential flow around a cylinder (without circulation) for which the solution is

$$f(z) \;=\; U\left\{z+\frac{R^2}{z}\right\} \tag{Bgeb9}$$

where U is the magnitude of the free stream velocity in the positive x direction and R is the radius of the cylinder whose center is at the origin. We will demonstrate that the conformal mapping

$$\zeta \;=\; z-\frac{R^2}{z} \tag{Bgec1}$$

transforms the cylinder in the $z = x + iy$ plane to a flat plate in the $\zeta = \xi + i\eta$ plane as sketched in Figure 3. Any point on the surface of the cylinder in the z plane is given by $z = Re^{i\theta}$. Consequently that surface appears in the ζ plane at

$$\zeta \;=\; Re^{i\theta}-Re^{-i\theta} \;=\; 2Ri\sin\theta \tag{Bgec2}$$

and is therefore entirely located on the η axis at a position given by $\eta = 2R\sin\theta$. The point on the surface of the cylinder at $\theta = \pi/2$ maps to $\xi = 0$, $\eta = 2R$ and the point on the surface of the cylinder at $\theta = -\pi/2$ maps to $\xi = 0$, $\eta = -2R$. Thus the cylinder has mapped into an infinitely thin flat plate of height $4R$ lying on the η axis. Furthermore, at very large values of z far from the cylinder the mapping reduces to $\zeta \approx z$ and therefore the flow in the ζ plane far from the origin is simply the same uniform stream of velocity U in the positive ξ direction. Hence the flow in the ζ plane is the flow of a uniform stream of velocity U past a flat plate of width $4R$ set normal to that oncoming stream. In this way we have found the solution for that potential flow, which can be presented parametrically as

$$f(z) \;=\; U\left\{z+\frac{R^2}{z}\right\} \quad \text{where} \quad \zeta \;=\; z-R^2/z \tag{Bgec3}$$

The velocity components in the ξ and η directions, namely u_ξ and u_η can then be computed using

$$\frac{df}{d\zeta} \;=\; u_\xi - iu_\eta \;=\; \frac{df}{dz}\frac{dz}{d\zeta} \;=\; U\left\{1-\frac{R^2}{z^2}\right\}\left\{1+\frac{R^2}{z^2}\right\}^{-1} \tag{Bgec4}$$

Notice that the velocities become infinite at the edges of the plate, that is at $\zeta = \pm 2iR$ (where $z = \pm iR$), since the denominator in equation (Bgec4) becomes zero at that location.

One of the more important potential flow results obtained using conformal mapping begins with the known solution for the flow past a circular cylinder (with circulation) and maps the circle into an airfoil shape using what is called the Joukowski mapping. This is described in detail in a following section.