Constitutive Relations for a Newtonian Fluid in Spherical Coordinates

The constitutive laws for a Newtonian fluid when written in spherical coordinates, (r, θ, ϕ), with velocities u_r, u_θ, u_ϕ in the r, θ, ϕ directions become:

$$\sigma_{rr} = -p + 2\mu\frac{\partial u_r}{\partial r} + \Lambda\left\{\frac{1}{r^2}\frac{\partial(r^2u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(u_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right\} \tag{Bhe1}$$

$$\sigma_{\theta\theta} = -p + 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}\right) + \Lambda\left\{\frac{1}{r^2}\frac{\partial(r^2u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(u_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right\} \tag{Bhe2}$$

$$\sigma_{\phi\phi} = -p + 2\mu\left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} + \frac{u_r}{r} + \frac{u_\theta\cot\theta}{r}\right) + \Lambda\left\{\frac{1}{r^2}\frac{\partial(r^2u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(u_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right\} \tag{Bhe3}$$

$$\sigma_{r\theta} = \sigma_{\theta r} = \mu\left(r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right) \tag{Bhe4}$$

$$\sigma_{r\phi} = \sigma_{\phi r} = \mu\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} + r\frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right)\right) \tag{Bhe5}$$

$$\sigma_{\theta\phi} = \sigma_{\phi\theta} = \mu\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi} + \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{u_\phi}{\sin\theta}\right)\right) \tag{Bhe6}$$

where μ is the dynamic viscosity and Λ is the second coefficient of viscosity. For a monatomic gas $\Lambda = -2\mu/3$. The above apply whether the fluid is compressible or incompressible. In the simple case of an *incompressible* fluid the constitutive relations become:

$$\sigma_{rr} = -p + 2\mu\frac{\partial u_r}{\partial r} \tag{Bhe7}$$

$$\sigma_{\theta\theta} = -p + 2\mu\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}\right) \tag{Bhe8}$$

$$\sigma_{\phi\phi} = -p + 2\mu\left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} + \frac{u_r}{r} + \frac{u_\theta\cot\theta}{r}\right) \tag{Bhe9}$$

$$\sigma_{r\theta} = \sigma_{\theta r} = \mu\left(r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right) \tag{Bhe10}$$

$$\sigma_{r\phi} = \sigma_{\phi r} = \mu\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} + r\frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right)\right) \tag{Bhe11}$$

$$\sigma_{\theta\phi} = \sigma_{\phi\theta} = \mu\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi} + \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{u_\phi}{\sin\theta}\right)\right) \tag{Bhe12}$$

where μ is the dynamic viscosity.