Shock Waves

It is appropriate to begin this section on normal shock waves by revisiting the piston thought experiment begun in section (Bod) in order to construct the events that result from a larger magnitude wave. To do so it is useful to construct a *space-time diagram* (an "x-t diagram") for the series of events that result from the motion of the piston (Figure 1). For convenience we assume that the motion of the piston begins

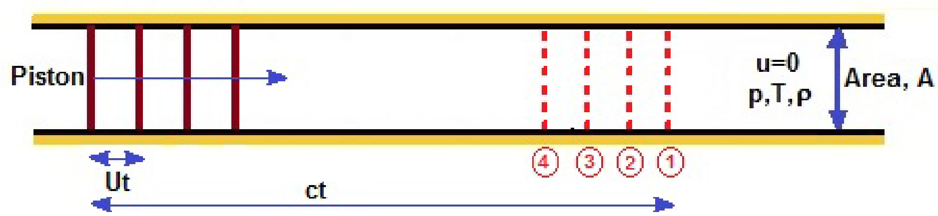

Figure 1: Tube with piston set in compressive motion at $t = 0$.

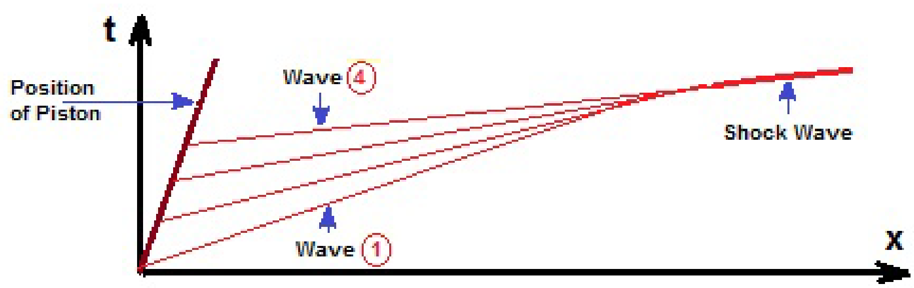

Figure 2: Space-time diagram for piston in compressive motion.

at time $t = 0$ and that, thereafter, it maintains a constant velocity, u; consequently it's motion in the space-time diagram is as shown in Figure 2. In the first instant of the motion a small compressive wave is generated and travels down the tube with velocity, c, as shown in Figure 2. However, as demonstrated in section (Bod) the gas behind that first wave will have a higher temperature and therefore a higher sound speed and so waves originating at the piston face a little later in time (as shown by waves 2, 3, 4) will travel progressively faster. Ultimately they will catch up with the waves ahead of them thus creating a larger wave from the accumulation of small waves (the process is not, of course discrete as depict in the diagram but continuous). The large amplitude wave that results from this accumulation is called a *shock wave*. One can visualize that it will travel faster than the first wave, faster than the speed of sound. Therefore it will not disperse since any small wave that might move ahead would have to travel through the gas at a slower velocity. It can also be shown by similar reasoning that no small fraction of the shock will break-off and get left behind for the same reasons.

The focus of this section will be on the conditions across a shock wave but for completeness and later reference we will briefly note what happens in the circumstance in which the piston is pulled backwards in an expansive rather than compressive motion as sketched in Figure 3. The space-time diagram for such an expansive motion is shown in Figure 4. The first wave still travels at the sound speed in the gas prior to the initiation of the motion. But the temperature behind that first wave is now less than that ahead of the first wave and so the second wave travels slower than the first wave. Subsequently waves travel progressively slower still and the result is a fan of waves (known as an *expansion fan*) that continue to disperse or spread apart as they progress downstream. Moreover the mean speed of the fan is significantly slower than the speed of sound.

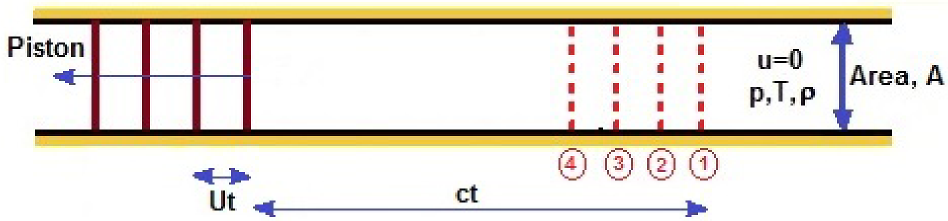

Figure 3: Tube with piston set in expansive motion at $t = 0$.

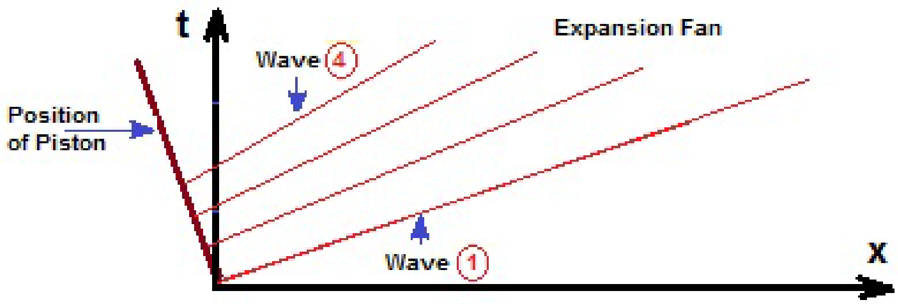

Figure 4: Space-time diagram for piston in expansive motion.

We will now proceed to determine the relations between the flow properties upstream of a shock wave (denoted by subscript 1) and those downstream of the shock (denoted by subscript 2). It is most convenient to conduct this investigation in a frame of reference fixed in the shock as shown in Figure 5. We anticipate

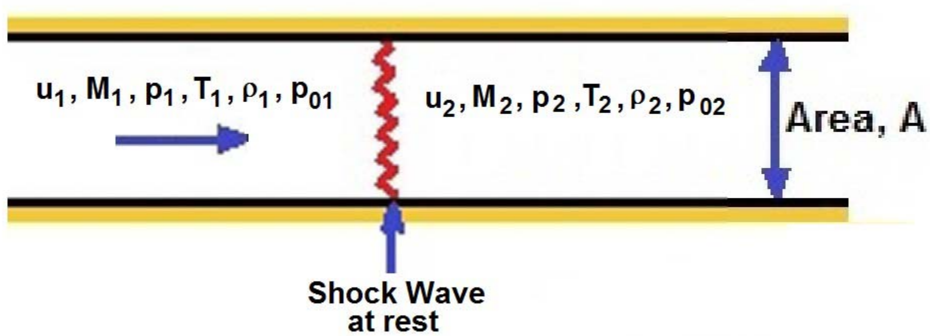

Figure 5: Frame fixed in a shock wave.

that the flow undergoes a non-isentropic process as it passes through the shock wave and therefore the conservation relations that need to be satisfied are those of mass, momentum and energy together with the equation of state (the perfect gas law in the present construction). These lead to the following relations between the fluid properties upstream and downstream of the shock:

- *Continuity*:

$$\rho_1 u_1 = \rho_2 u_2 \qquad \text{(Boh1)}$$

since the shock is assume infinitely thin and therefore the area of the flow is the same on both sides.

- *Energy*:

$$c_pT_1 + u_1^2/2 \;=\; c_pT_2 + u_2^2/2 \;=\; c_pT_0 \tag{Boh2}$$

where T_0 is the stagnation temperature which must be the same on both sides of the shock.

- *Momentum*:

$$p_1 - p_2 \;=\; \rho_2 u_2^2 - \rho_1 u_1^2 \;=\; \rho_1 u_1 (u_2 - u_1) \tag{Boh3}$$

using continuity.

- *State*:

$$\frac{p_1}{\rho_1 T_1} \;=\; \frac{p_2}{\rho_2 T_2} \tag{Boh4}$$

- *and by definition*:

$$M_1^2 \;=\; \frac{u_1^2}{\gamma \mathcal{R} T_1} \qquad \text{and} \qquad M_2^2 \;=\; \frac{u_2^2}{\gamma \mathcal{R} T_2} \tag{Boh5}$$

The momentum and continuity equations lead to

$$u_1 u_2 (u_2 - u_1) \;=\; \mathcal{R}(u_2 - u_1)\left\{T + \frac{u_1 u_2}{2c_p}\right\} \tag{Boh6}$$

This means that *either* $u_1 = u_2$ and there is no shock since all the flow properties are the same across the shock *or*

$$u_1 u_2 \;=\; \mathcal{R}\left\{T + \frac{u_1 u_2}{2c_p}\right\} \;=\; \frac{2\gamma \mathcal{R} T_0}{(\gamma+1)} \tag{Boh7}$$

This is known as Prandtl's equation for a shock wave. After some algebra, it leads to the relation between the Mach numbers on either side of a shock:

$$M_1 M_2 \;=\; \frac{2}{(\gamma+1)}\left[1 + \frac{(\gamma-1)}{2}M_1^2\right]^{\frac{1}{2}}\left[1 + \frac{(\gamma-1)}{2}M_2^2\right]^{\frac{1}{2}} \tag{Boh8}$$

or

$$M_2^2 \;=\; \frac{2 + (\gamma+1)M_1^2}{2\gamma M_1^2 - (\gamma-1)} \tag{Boh9}$$

The resulting downsream Mach numbers, M_2, are plotted in Figure 6 and tabulated in Figure 9 as a function of the upstream Mach number, M_1. Relations between the other properties upstream and downstream of the shock also follow from the basic equations (Boh1) through (Boh5) once M_2 has been determined:

$$\frac{T_2}{T_1} \;=\; \frac{1 + \frac{(\gamma-1)}{2}(1 - M_1^2)}{1 + \frac{(\gamma-1)}{2}(1 - M_2^2)} \tag{Boh10}$$

$$\frac{p_2}{p_1} \;=\; \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2} \qquad \text{and} \qquad \frac{\rho_2}{\rho_1} \;=\; \frac{T_1}{T_2}\frac{p_2}{p_1} \tag{Boh11}$$

It is also useful to derive the relation between the stagnation pressures on the two sides of the shock, p_{01} and p_{02}, where

$$p_{01} \;=\; p_1\left[1 + \frac{(\gamma-1)}{2}M_1^2\right]^{\gamma/(\gamma-1)} \qquad \text{and} \qquad p_{02} \;=\; p_2\left[1 + \frac{(\gamma-1)}{2}M_2^2\right]^{\gamma/(\gamma-1)} \tag{Boh12}$$

The ratio p_{02}/p_{01} is plotted as a fuction of M_1 in Figure 6 which demonstrates that the stagnation pressure always decreases across a shock. The other ratios, p_2/p_1, T_2/T_1, and ρ_2/ρ_1 are plotted against M_1 in Figure 7 and all of these ratios are tabulated in Figure 9 for a range of upstream Mach numbers.

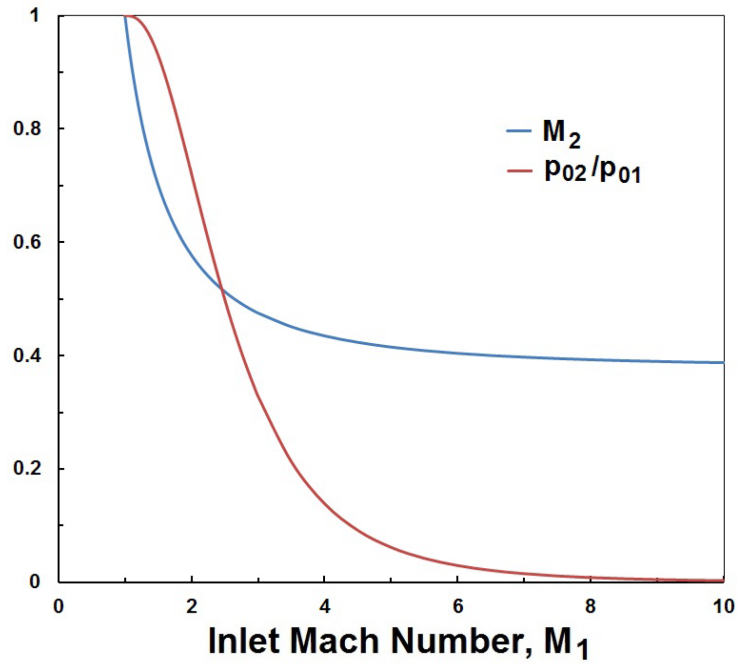

Figure 6: Graphs of M_2 and p_{02}/p_{01} against inlet Mach number, M_1.

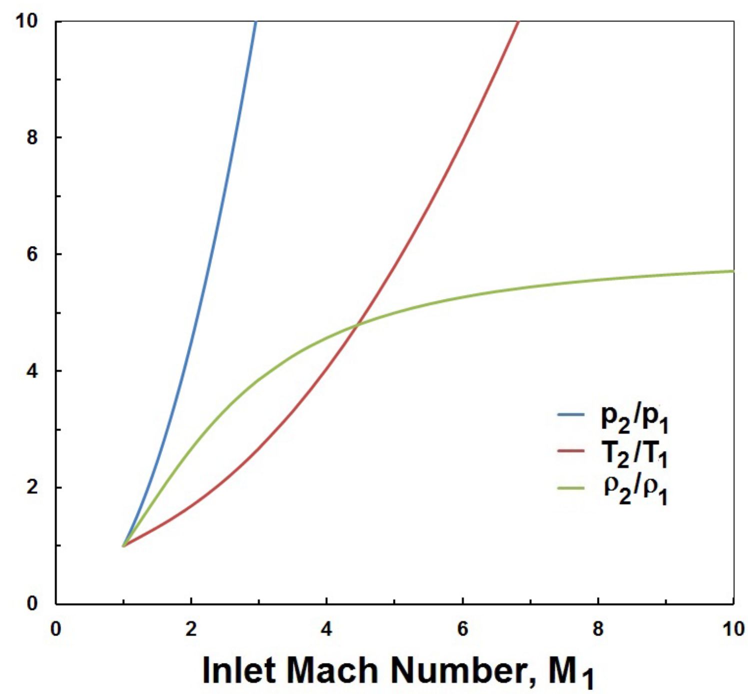

Figure 7: Graphs of p_2/p_1, T_2/T_1 and ρ_2/ρ_1 against inlet Mach number, M_1.

In addition, it is important to evaluate the entropy change across the shock which can be obtained by integrating the basic thermodynamic relation:

$$T\ ds \ = \ c_v\ dT + pd\left(\frac{1}{\rho}\right) \tag{Boh13}$$

to obtain

$$\frac{(s_2 - s_1)}{c_v} = \ln\left(\frac{T_2}{T_1}\right) - (\gamma - 1)\ln\left(\frac{\rho_2}{\rho_1}\right) \quad \text{(Boh14)}$$

This quantity, $(s_2 - s_1)/c_v$, is plotted against M_1 in Figure 8. Note that the entropy increases for a shock with $M_1 > 1$ but decreases for $M_1 < 1$. It follows from the second law of thermodynamics that it is not possible to have a shock with a subsonic upstream flow. Shocks only occur with supersonic upstream conditions and therefore subsonic downstream flow.

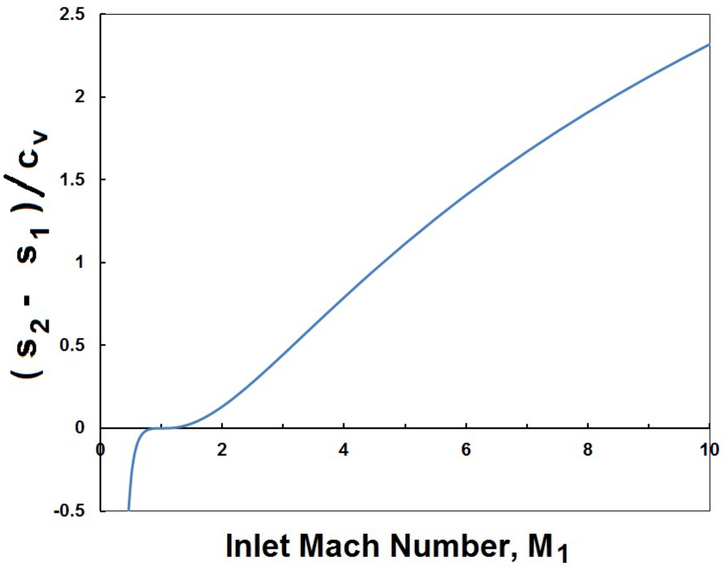

Figure 8: Graph of $(s_2 - s_1)/c_v$ as a function of the inlet Mach number, M_1.

As was discussed in the introduction to this section, a shock wave is self-steepening in the sense that any small waves that might possibly detach from the upstream or downstream sides of the shock propagate back into the shock since the wave velocity ahead of the shock is less than the shock speed while the wave velocity behind the shock is greater than the shock speed. Then, one remaining question is what determines the thickness, ϵ, of the shock? To answer this, recall that the non-isentropic character of a shock wave is a result of the irreversible changes that occur within its structure as kinetic energy is converted to heat. The mechanism for this conversion is the action of viscosity and therefore the thickness of a shock is necessarily proportional to the viscosity, μ, of the fluid. Since the only other quantities that might determine ϵ are the fluid density, ρ, and the strength of the shock, $\Delta p = p_2 - p_1$ dimensional analysis requires that

$$\epsilon \propto \frac{\mu}{(\rho \Delta p)^{\frac{1}{2}}} \quad \text{(Boh15)}$$

The actual thickness of a shock wave in air is very small indeed, of the order of the molecular mean free path and therefore requires more detailed analysis than can be followed here.

Shock waves occur in a myriad of different contexts, a few of which will be discussed in the sections that follow. They represent a mechanism by which a supersonic flow can transition to a subsonic flow and they consequently occur where the boundary conditions necessitate such a transition. In the next section we explore one such context.

M_1	M_2	p_2/p_1	T_2/T_1	ρ_2/ρ_1	p_{02}/p_{01}
1	1	1	1	1	1
1.05	0.9531	1.1196	1.0328	1.084	0.9999
1.1	0.9118	1.245	1.0649	1.1691	0.9989
1.15	0.875	1.3763	1.0966	1.255	0.9967
1.2	0.8422	1.5133	1.128	1.3416	0.9928
1.25	0.8126	1.6563	1.1594	1.4286	0.9871
1.3	0.786	1.805	1.1909	1.5157	0.9794
1.35	0.7618	1.9596	1.2226	1.6028	0.9697
1.4	0.7397	2.12	1.2547	1.6897	0.9582
1.45	0.7196	2.2863	1.2872	1.7761	0.9448
1.5	0.7011	2.4583	1.3202	1.8621	0.9298
1.55	0.6841	2.6363	1.3538	1.9473	0.9132
1.6	0.6684	2.82	1.388	2.0317	0.8952
1.65	0.654	3.0096	1.4228	2.1152	0.876
1.7	0.6405	3.205	1.4583	2.1977	0.8557
1.75	0.6281	3.4063	1.4946	2.2791	0.8346
1.8	0.6165	3.6133	1.5316	2.3592	0.8127
1.85	0.6057	3.8263	1.5693	2.4381	0.7902
1.9	0.5956	4.045	1.6079	2.5157	0.7674
1.95	0.5862	4.2696	1.6473	2.5919	0.7442
2	0.5774	4.5	1.6875	2.6667	0.7209
2.05	0.5691	4.7363	1.7285	2.74	0.6975
2.1	0.5613	4.9783	1.7705	2.8119	0.6742
2.15	0.554	5.2263	1.8132	2.8823	0.6511
2.2	0.5471	5.48	1.8569	2.9512	0.6281
2.25	0.5406	5.7396	1.9014	3.0186	0.6055
2.3	0.5344	6.005	1.9468	3.0845	0.5833
2.35	0.5286	6.2763	1.9931	3.149	0.5615
2.4	0.5231	6.5533	2.0403	3.2119	0.5401
2.45	0.5179	6.8363	2.0885	3.2733	0.5193
2.5	0.513	7.125	2.1375	3.3333	0.499
2.55	0.5083	7.4196	2.1875	3.3919	0.4793
2.6	0.5039	7.72	2.2383	3.449	0.4601
2.65	0.4996	8.0263	2.2902	3.5047	0.4416
2.7	0.4956	8.3383	2.3429	3.559	0.4236
2.75	0.4918	8.6563	2.3966	3.6119	0.4062
2.8	0.4882	8.98	2.4512	3.6636	0.3895
2.85	0.4847	9.3096	2.5067	3.7139	0.3733
2.9	0.4814	9.645	2.5632	3.7629	0.3577
2.95	0.4782	9.9863	2.6206	3.8106	0.3428
3	0.4752	10.333	2.679	3.8571	0.3283
3.5	0.4512	14.125	3.3151	4.2609	0.2129
4	0.435	18.5	4.0469	4.5714	0.1388
4.5	0.4236	23.458	4.8751	4.8119	0.0917
5	0.4152	29	5.8	5	0.0617
5.5	0.409	35.125	6.8218	5.1489	0.0424
6	0.4042	41.833	7.9406	5.2683	0.0297
6.5	0.4004	49.125	9.1564	5.3651	0.0211
7	0.3974	57	10.469	5.4444	0.0154
7.5	0.3949	65.458	11.879	5.5102	0.0113
8	0.3929	74.5	13.387	5.5652	0.0085
8.5	0.3912	84.125	14.991	5.6117	0.0064
9	0.3898	94.333	16.693	5.6512	0.005
9.5	0.3886	105.13	18.492	5.685	0.0039
10	0.3876	116.5	20.388	5.7143	0.003

Figure 9: Tabulated values of M_2, p_2/p_1, T_2/T_1, ρ_2/ρ_1 and p_{02}/p_{01} against the inlet Mach number, M_1, for a normal shock wave with $\gamma = 1.4$.