One-Dimensional Isentropic Flow

Consider the relations that govern the one-dimensional steady, isentropic flow of a perfect gas in any pipe, duct or streamtube as sketched in Figure 1. The assumption of isentropic flow necessarily implies (1) that there is no heat addition to or removal from the flow and (2) that the flow is assumed to be thermodynamically reversible and therefore that all viscous or other irreversible effects are being neglected. We will return later to investigate both viscous effects and heat exchange effects in a real pipe or duct flow. The following conservation laws must apply to this steady one-dimensional flow:

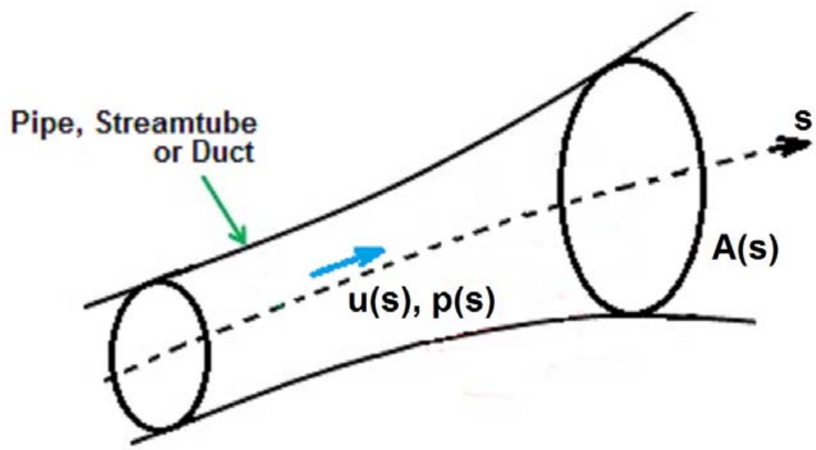

Figure 1: Any pipe, duct or streamtube.

- Conservation of mass requires that

$$\rho u A \;=\; \text{constant} \tag{Bob1}$$

where $A(s)$ is the cross-sectional area of the streamtube (a function of the coordinate, s, measured along the centerline of the streamtube) and $\rho(s)$ and $u(s)$ are the density and velocity of the flow averaged over that cross-section. Differentiating this leads to

$$\rho u dA + uAd\rho + \rho A du \;=\; 0 \tag{Bob2}$$

- The one-dimensional momentum equation requires that

$$\rho u \frac{\partial u}{\partial s} \;=\; -\frac{\partial p}{\partial s} \;-\; \rho g \frac{\partial z}{\partial s} \tag{Bob3}$$

where $p(s)$ and $z(s)$ are respectively the pressure and elevation averaged over the cross-section, $A(s)$. Note for comparison that the corresponding incompressible flow in which ρ is a constant allows equation (Bob3) to be written as

$$\frac{\partial}{\partial s}\left\{p + \frac{\rho u^2}{2} + \rho g z\right\} \;=\; 0 \tag{Bob4}$$

which leads to Bernoulli's equation that states that the quantity in parentheses is a constant along any streamline. However, for the compressible flow that is the subject of this section equation (Bob3) leads to

$$\rho u du + \rho g dz + dp \;=\; 0 \tag{Bob5}$$

- Since the flow is isentropic it follows from the thermodynamic definition of the specific enthalpy, h, that

$$\frac{dp}{\rho} = dh \tag{Bob6}$$

Combining equations (Bob5) and (Bob6) it follows that in steady, isentropic, one-dimensional flow

$$d\left\{\frac{u^2}{2} + gz + h\right\} = dh^* = 0 \tag{Bob6}$$

and therefore the total enthalpy, h^*, is constant along any streamline in such flow. Note that in most of the high speed flows with which we will be concerned in these sections, the gravitational term, gz, is negligible and will be omitted.

- For a perfect gas we also have

$$p = \rho \mathcal{R} T \qquad \text{and} \qquad h = c_p T + \text{constant} \tag{Bob7}$$

and the isentropic relations

$$p \propto T^{\frac{\gamma}{(\gamma-1)}} \quad ; \quad p \propto \rho^{\gamma} \quad ; \quad \rho \propto T^{\frac{1}{(\gamma-1)}} \tag{Bob8}$$

As an example of the application of these relations, consider the steady. isentropic flow around an airplane in a frame of reference fixed in the plane. Equation (Bob6) for a perfect gas without the gravitational term becomes

$$c_p T + \frac{u^2}{2} = \text{constant} \tag{Bob9}$$

along any streamline. Consider, therefore the streamline connecting the flow far upstream of the plane (where the velocity is $u = U_\infty$ and the temperature is denoted by T_∞) with the stagnation point on the nose of the plane where $u = 0$ and the temperature is denoted by T_S. From equation (Bob9) it follows that

$$\frac{T_S}{T_\infty} = 1 + \frac{U^2}{2c_p T\infty} = 1 + \frac{(\gamma-1)}{2}\frac{U^2}{\gamma \mathcal{R} T\infty} \tag{Bob10}$$

We shall see in the section which follows that $\gamma \mathcal{R} T$ is the square of the speed of sound, c, so that equation (Bob10) becomes

$$\frac{T_S}{T_\infty} = 1 + \frac{(\gamma-1)}{2} M_\infty^2 \tag{Bob11}$$

where M_∞ is the Mach number of the plane, U/c.

For numerical examples we take air whose $\mathcal{R} = 280 \ m^2/s^2 K^o$, $\gamma = 1.4$ and $c_p = 980 \ m^2/s^2 K^o$. Then at $u \approx 300 m/s$ equation (Bob11) yields a temperature difference between the stagnation point and far upstream, $T_S - T_\infty$ of about $50^o K$. An aluminium fuselage can withstand such a temperature. Even at a speed of $500 m/s$ the temperature difference is a manageable $120^o K$. But at $1500 m/s$ the temperature difference becomes $1000^o K$ and an aluminum skin is no longer possible. Airplanes at those kinds of speeds require titanium or carbon fiber skins.