Energy Equation

In this section we provide details on the energy equation which is a keystone for the material in the subject of compressible flows. It follows directly from the first law of thermodynamics (see section (Acc)); attention will be given to both reversible flows and irreversible flows.

The first law applies to a particular Lagrangian mass of fluid and so we begin by writing the first law in the form

$$\frac{DQ}{Dt} + \frac{DW}{Dt} = \frac{DE^*}{Dt} \tag{Boc1}$$

where D/Dt is the Lagrangian time derivative that follows the fluid, Q is the heat added to that Lagrangian volume, W is the work done on the Lagrangian volume and E^* is the *total* stored energy in the Lagrangian volume. We will assess and evaluate each of the three terms in equation (Boc1) below but first several preliminary notes will aid that assessment:

- In evaluating the terms in equation (Boc1) we will use the transport theorem (see section (Bae)) which states that for any transportable property, B:

$$\frac{D}{Dt}\int_V BdV = \int_{V_0} \frac{\partial B}{\partial t}\, dV_0 + \int_{S_0} B\underline{u}\cdot\underline{n}\, dS_0 \tag{Boc2}$$

  where V is an arbitrary Lagrangian volume and V_0 and S_0 are the Eulerian volume and surface of V at the moment of evaluation of the Lagrangian time derivative, D/Dt ($\underline{n}$ is the outward unit normal to the surface S_0). Using Gauss' theorem which says that for any vector quantity, say $\underline{X}$:

$$\int_{S_0} \underline{X}\cdot\underline{n}\, dS_0 = \int_{V_0} \nabla\cdot\underline{X}\, dV_0 = \int_{V_0} \frac{\partial X_j}{\partial x_j}\, dV_0 \tag{Boc3}$$

  we may write equation (Boc2) as

$$\frac{D}{Dt}\int_V BdV = \int_{V_0} \left\{ \frac{\partial B}{\partial t} + \frac{\partial (Bu_j)}{\partial x_j} \right\} dV_0 \tag{Boc4}$$

  As an example of the use of equation (Boc4) we note that the equation of conservation of mass may be obtained by setting $B=\rho$ and recognizing that mass conservation requires that the Lagrangian derivative of the mass in the volume V must be zero so that the left-hand side of equation (Boc4) must be zero and therefore

$$\int_{V_0} \left\{ \frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_j)}{\partial x_j} \right\} dV_0 = 0 \tag{Boc5}$$

  and therefore, since V was arbitrarily chosen the integrand must be zero so that

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_j)}{\partial x_j} = 0 \tag{Boc6}$$

  which is, of course, the equation of continuity for a compressible fluid.

- The other clarification that is needed is to ensure that the quantity, E^*, in equation (Boc1) includes all the forms of storable energy within the Lagrangian volume, V. Thus

$$E^* = \int_{V_0} \left\{ e + \frac{1}{2}|u|^2 + gz \right\} dV_0 \tag{Boc5}$$

where each of the terms in the integrand represent respectively the contributions to the total stored energy of the internal energy, the kinetic energy and the gravitational potential energy. It is, of course, possible to include other forms of stored energy such as chemical energy, but the sum in equation (Boc5) is sufficient for our purposes. Then using the transport theorem, equation (Boc4), to convert to the equation for the Eulerian volume, V_0, we obtain

$$\frac{DE^*}{Dt} = \int_{V_0} \frac{\partial}{\partial t}\left\{e + \frac{1}{2}|u|^2 + gz\right\} + \frac{\partial}{\partial x_j}\left\{\rho u_j (e + \frac{1}{2}|u|^2 + gz)\right\} \, dV_0 \tag{Boc6}$$

or, expanding and using the continuity equation to delete some terms,

$$\frac{DE^*}{Dt} = \int_{V_0} \rho\frac{\partial}{\partial t}\left\{e + \frac{1}{2}|u|^2 + gz\right\} + \rho u_j \frac{\partial}{\partial x_j}\left\{e + \frac{1}{2}|u|^2 + gz\right\} \, dV_0 \tag{Boc7}$$

- Note that the term DQ/Dt in equation (Boc1) represents the rate of heat addition to the Lagrangian volume by the surrounding fluid, perhaps by conduction or radiation. Since these are both irreversible processes it follows that $DQ/Dt = 0$ for a reversible, isentropic process. For an irreversible process in which the heat added to the volume per unit mass is denoted by q it follows that

$$\frac{DQ}{Dt} = \frac{D}{Dt}\left\{\int_V \rho q dV\right\} = \int_{V_0}\left\{\frac{\partial(\rho q)}{\partial t} + \frac{\partial(\rho u_j q)}{\partial x_j}\right\} \, dV_0 \tag{Boc8}$$

  and expanding and using the continuity equation this becomes

$$\frac{DQ}{Dt} = \int_{V_0} \rho\left\{\frac{\partial q}{\partial t} + u_j\frac{\partial q}{\partial x_j}\right\} \, dV_0 = \int_{V_0} \rho\frac{Dq}{Dt} \, dV_0 \tag{Boc9}$$

  where the second term in the integrand becomes zero if the heat addition were uniform as it sometimes can be assumed to be. Note again that this is only non-zero for an irreversible process.

- Note that the term DW/Dt in equation (Boc1) represents the rate of work done on the Lagrangian volume by the surrounding fluid plus the rate of work done by other moving parts within the Lagrangian volume. The first contribution, namely the work done on the Lagrangian volume by the surrounding fluid, more specifically by the pressure, p, in the surrounding fluid, will be given by

$$-\int_{S_0} p\underline{u}\cdot\underline{n} \, dS_0 = -\int_{V_0}\frac{\partial(pu_j)}{\partial x_j} \, dV_0 = \int_{V_0}\left\{\frac{\partial p}{\partial t} - \rho\frac{D(p/\rho)}{Dt}\right\} \, dV_0 \tag{Boc10}$$

  using Gauss' theorem and the equation of continuity. Adding to this the rate of work done by external, mechanical means denoted by $\dot{w}_E$ per unit mass we can write

$$\frac{DW}{Dt} = \int_{V_0}\left[\left\{\frac{\partial p}{\partial t} - \rho\frac{D(p/\rho)}{Dt}\right\} + \rho\dot{w}_E\right] \, dV_0 \tag{Boc11}$$

  Parenthetically we note that, for simplicity and because we will not use them, we have not included the work done by the viscous stresses acting on the surface of the volume.

Finally, having converted all the terms in equation (Boc1) to evaluations in the Eulerian volume, V_0, the first law can be written as

$$\int_{V_0}\rho\frac{D}{Dt}\left\{e + \frac{1}{2}|u|^2 + gz\right\} \, dV_0 = \int_{V_0}\rho\left\{\frac{Dq}{Dt} + \dot{w}_E + \frac{1}{\rho}\frac{\partial p}{\partial t} - \frac{D(p/\rho)}{Dt}\right\} \, dV_0 \tag{Boc12}$$

Since the volume, V_0, is arbitrary we can drop the integral signs and combine terms so that the first law becomes

$$\frac{D}{Dt}\left\{\frac{p}{\rho}+e+\frac{1}{2}|u|^2+gz\right\} = \frac{Dh^*}{Dt} = \frac{Dq}{Dt}+\dot{w}_E+\frac{1}{\rho}\frac{\partial p}{\partial t} \tag{Boc13}$$

Thus, after all this manipulation, we come to a very simple conclusion, namely that the first law in a flowing fluid can be expressed as follows: the Lagrangian rate of change with time of the specific total enthalpy, h^*, along any streamline is equal to the Lagrangian rate of addition of heat plus the rate of external work done on the fluid plus a term involving the pressure that is zero for steady flow. In a steady, reversible or isentropic flow the entire right hand side is zero and the energy equation simply becomes that the total enthalpy is constant along all streamlines, a result we obtained much more readily in the preceding section. What the above analysis has done is to identify the form of the first law that should be deployed in an irreversible flow. In a steady irreversible flow the first law becomes

$$\frac{Dh^*}{Dt} = \frac{Dq}{Dt}+\dot{w}_E \tag{Boc14}$$

Often either the external work done and/or the heat addition is zero and the first law becomes even simpler.

The common practical application of the form, equation (Boc14), of the first law for a steady flow is for the kind of flow sketched in Figure 1 comprised of a duct with a mass flow rate through it denoted by $\dot{m}$ as well as a rate of heat input denoted by $\dot{Q}$ and a rate of work done $\dot{W}_E$. Then the result given in

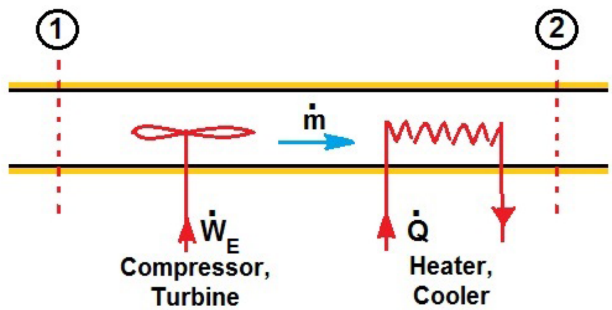

Figure 1: Duct with heat addition/removal and/or work done.

equation (Boc14) means that the total specific enthalpy entering the duct at station 1, h_1^*, and that exiting at station 2, h_2^*, will be related by

$$h_2^*-h_1^* = \frac{(\dot{Q}+\dot{W}_E)}{\dot{m}} \tag{Boc15}$$