Laminar Boundary Layer Equations

In this section we will develop the appropriate versions of the equations of motion for the flow in a laminar

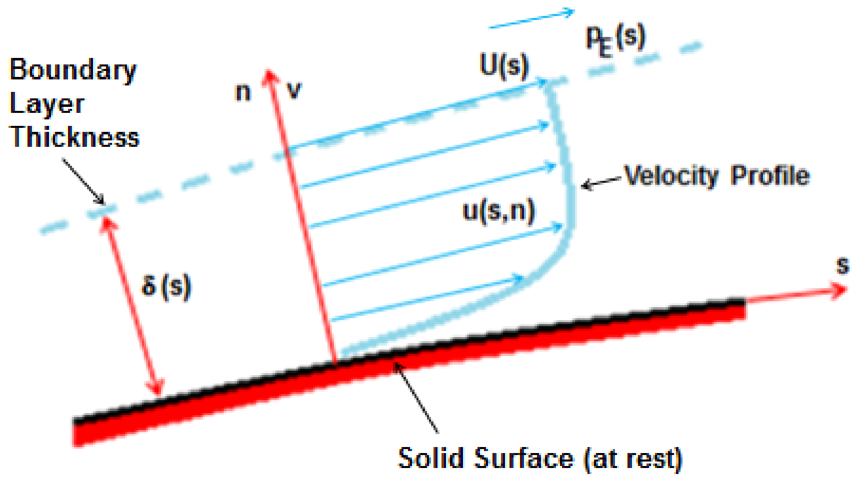

Figure 1: Boundary layer in a planar flow.

boundary layer of an incompressible, Newtonian fluid of constant and uniform density and viscosity. For simplicity we focus initially on steady, planar flow. It is necessarily assumed that the Reynolds number, $Re_L = UL/\nu$, $s \leq L$, (where L is a typical dimension of the body on which the boundary layer develops) is much greater than unity for otherwise, as discussed in the preceding section (Bja), there would be no such boundary layer; we explore the consequences of this restriction, $Re_L >> 1$, once the solution to the flow has been obtained. It is assumed that the flow external to the boundary layer is known *a priori* so that $U(s)$ and $p_E(s)$ are known.

To proceed we first note that, if the qualitative evaluation of the boundary layer thickness, $\delta(s)$, contained in section (Bja) is correct (we will have to confirm this once this more precise evaluation has been concluded), namely that

$$\delta(s) \approx (\nu s/U)^{1/2} \qquad \text{(Bjb1)}$$

then it follows that the angle that the edge of the boundary layer makes with the solid surface is given by

$$\frac{d\delta}{ds} \approx \left(\frac{\nu}{Us}\right)^{1/2} = \left(\frac{1}{Re_s}\right)^{1/2} \qquad \text{(Bjb2)}$$

where $Re_s = Us/\nu$ is the Reynolds number based on U and the distance measured along the surface from the point where the boundary layer began, s. Clearly then, if $Re_L >> 1$ and $s < L$ it follows that $d\delta/ds$ is very small except very close to $s = 0$. Therefore, the velocity vector in the boundary layer must be nearly parallel with the solid surface and the order of magnitude of v/u must be small and given by $d\delta/ds$ or $Re_s^{-1/2}$. This permits an evaluation of the order of magnitude of the various terms in the equations governing the flow in the boundary layer. This, in turn, allows development of approximations to those equations known as the *boundary layer equations*. Exact and approximate solutions to those boundary layer equations will be the focus of next few sections. To proceed with the evaluation of the relative magnitude of the various terms in those equations of motion for the flow in a boundary layer it is convenient to denote the small quantity $d\delta/ds$ by ϵ. Then the magnitudes of u and v are respectively U and ϵU or

$$u = O(U) \quad \text{and} \quad v = O(\epsilon U) \tag{Bjb3}$$

Furthermore since $\delta << L$ we neglect the effects of curvature in the definition of the (s, n) coordinate system and can therefore write the equations of continuity and motion (equations (Bcd7), (Bhf8) and (Bhf9)) for this steady flow as

$$\frac{\partial u}{\partial s} + \frac{\partial v}{\partial n} = 0 \tag{Bjb4}$$

$$\rho\left\{u\frac{\partial u}{\partial s} + v\frac{\partial u}{\partial n}\right\} = -\frac{\partial p}{\partial s} + \mu\left\{\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial n^2}\right\} \tag{Bjb5}$$

$$\rho\left\{u\frac{\partial v}{\partial s} + v\frac{\partial v}{\partial n}\right\} = -\frac{\partial p}{\partial n} + \mu\left\{\frac{\partial^2 v}{\partial s^2} + \frac{\partial^2 v}{\partial n^2}\right\} \tag{Bjb6}$$

where we have absorbed the assumed-conservative body force terms into the pressure terms so that p is now defined as the pressure minus the gravitational contribution.

In order to evaluate the derivatives in these equations we note that derivatives in the n direction will scale like $1/\delta$ and that

$$\frac{\partial}{\partial n} = O\left(\frac{1}{\delta}\right) \tag{Bjb7}$$

The evaluations (Bjb3) and (Bjb7) coupled with equation (Bjb4) mean that

$$\frac{\partial v}{\partial n} = O\left(\frac{\epsilon U}{\delta}\right) \quad \text{and} \quad \frac{\partial u}{\partial s} = O\left(\frac{\epsilon U}{\delta}\right) \tag{Bjb8}$$

and therefore

$$\frac{\partial}{\partial s} = O\left(\frac{\epsilon}{\delta}\right) \tag{Bjb9}$$

In other words the derivatives in the s direction are order ϵ smaller than derivatives in the n direction.

With these orders of magnitude in mind we are equipped to evaluate the order of magnitudes of the various terms involving velocity in the equations of motion (Bjb5) and (Bjb6). First equation (Bjb5), the s-momentum equation, can be written as:

$$\begin{array}{ccccccccc} u\dfrac{\partial u}{\partial s} & + & v\dfrac{\partial u}{\partial n} & = & -\dfrac{1}{\rho}\dfrac{\partial p}{\partial s} & + & \nu\dfrac{\partial^2 u}{\partial s^2} & + & \nu\dfrac{\partial^2 u}{\partial n^2} \\ \Uparrow & & \Uparrow & & & & \Uparrow & & \Uparrow \\ O\left(U\dfrac{\epsilon U}{\delta}\right) & & O\left(U\dfrac{\epsilon U}{\delta}\right) & & & & O\left(\dfrac{\nu\epsilon^2 U}{\delta^2}\right) & & O\left(\dfrac{\nu U}{\delta^2}\right) \end{array}$$

Consequently, the penultimate term can be neglected compared with the last term and this constitutes the first approximation incorporated in the boundary layer equations. Evidently the first two terms appear to be of the same order of magnitude and both should be retained leaving the first of the boundary layer equations as

$$u\frac{\partial u}{\partial s} + v\frac{\partial u}{\partial n} = -\frac{1}{\rho}\frac{\partial p}{\partial s} + \nu\frac{\partial^2 u}{\partial n^2} \tag{Bjb10}$$

A similar treatment of the second equation of motion (Bjb6), the n-momentum equation, yields

$$\begin{array}{ccccccccc} u\dfrac{\partial v}{\partial s} & + & v\dfrac{\partial v}{\partial n} & = & -\dfrac{1}{\rho}\dfrac{\partial p}{\partial n} & + & \nu\dfrac{\partial^2 v}{\partial s^2} & + & \nu\dfrac{\partial^2 v}{\partial n^2} \\ \Uparrow & & \Uparrow & & & & \Uparrow & & \Uparrow \\ O\left(U\dfrac{\epsilon^2 U}{\delta}\right) & & O\left(U\dfrac{\epsilon^2 U}{\delta}\right) & & & & O\left(\dfrac{\nu\epsilon^3 U}{\delta^2}\right) & & O\left(\dfrac{\nu U\epsilon}{\delta^2}\right) \end{array}$$

Note that all of the velocity terms in this n-momentum equation are ϵ smaller than the corresponding terms in the s-momentum equation. Therefore we argue that the gradients in the pressure in the n direction are small compared with the gradients in the pressure in the s direction and the second of the boundary layer approximations is to neglect all the velocity terms in the n-momentum equation and write simply that

$$\frac{\partial p}{\partial n} = 0 \tag{Bjb11}$$

so that, within the boundary layer, p varies only with s and must therefore be equal to the known pressure, $p_E(s)$, in the exterior flow at the edge of the boundary layer. Therefore, within the boundary layer $p(s,n) = p_E(s)$ and therefore the term $\partial p/\partial s = dp_E/ds$ in the first boundary layer equation is a known input function.

In summary, the problem of solving for the flow in a planar, incompressible, Newtonian boundary layer involves solving the equations

$$\frac{\partial u}{\partial s} + \frac{\partial v}{\partial n} = 0 \tag{Bjb12}$$

$$u\frac{\partial u}{\partial s} + v\frac{\partial u}{\partial n} = -\frac{1}{\rho}\frac{dp_E(s)}{ds} + \nu\frac{\partial^2 u}{\partial n^2} \tag{Bjb13}$$

for $u(s,n)$ and $v(s,n)$ given the pressure distribution, $p_E(s)$. An alternative would be to solve the single equation

$$\frac{\partial\psi}{\partial n}\frac{\partial^2\psi}{\partial s\partial n} - \frac{\partial\psi}{\partial s}\frac{\partial^2\psi}{\partial n^2} = -\frac{1}{\rho}\frac{dp_E(s)}{ds} + \nu\frac{\partial^3\psi}{\partial n^3} \tag{Bjb14}$$

for the streamfunction, $\psi(s,n)$, where to satisfy the continuity equation (Bjb12) we must have

$$u = \frac{\partial\psi}{\partial n} \qquad \text{and} \qquad v = -\frac{\partial\psi}{\partial s} \tag{Bjb15}$$

The boundary conditions that these solutions must satisfy are:

1. At the solid surface, the no-slip condition and the zero normal velocity condition must apply and since the problem has been posed in a frame of reference fixed in that solid:

$$(u)_{n=0} = (\frac{\partial\psi}{\partial n})_{n=0} = 0 \tag{Bjb16}$$

$$(v)_{n=0} = (\frac{\partial\psi}{\partial s})_{n=0} = 0 \qquad \text{or} \qquad (\psi)_{n=0} = \text{Constant usually zero} \tag{Bjb17}$$

2. At the edge of the boundary layer the velocity, u, must approach the velocity U. As will be seen, this condition is applied in two alternative forms:

$$(u)_{n\to\delta} \to U(s) \qquad \text{or} \qquad (u)_{n\to\infty} \to U(s) \tag{Bjb18}$$

Finally we observe that in most cases investigated herein, the external flow is irrotational and therefore by Bernoulli's equation,

$$p_E(s) + \frac{1}{2}\rho U(s)^2 \quad = \quad \text{Constant} \qquad \text{and} \qquad \frac{1}{\rho}\frac{dp_E}{ds} \quad = \quad -U\frac{dU}{ds} \tag{Bjb19}$$

Consequently, in such cases, the boundary layer equations for steady, planar, incompressible flow become

$$\frac{\partial u}{\partial s} + \frac{\partial v}{\partial n} = 0 \tag{Bjb20}$$

$$u\frac{\partial u}{\partial s} + v\frac{\partial u}{\partial n} = U\frac{dU}{ds} + \nu\frac{\partial^2 u}{\partial n^2} \tag{Bjb21}$$