Slender body theory

The objective of slender-body theory is to take advantage of the slenderness in order to achieve simplifications in obtaining approximate solutions for the flow around such bodies. The development of low-Reynolds-number slender-body theory evolved through the work of Burgers (1938), Broersma 1196O, and Tuck (1964); later work by Taylor (1969), Tillett (1970), Batchelor (1970a), Cox (1970), and Blake (1974b) concentrated on construction of slender-body solutions by distributions of fundamental singularities along an axis of the body. We note that with the exception of Batchelor's (1970a) work on arbitrary cross-section, researchers have concentrated on bodies of circular cross-section.

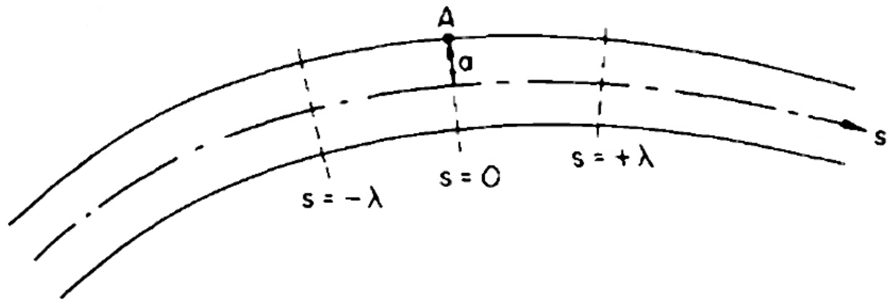

Figure 1: Slender-body schematic.

In choosing axes fixed relative to a particular section of the slender body under examination (Figure 1), we will seek the distribution of stokeslets, doublets, etc. on the axis of the body that will satisfy the no-slip condition at points such as A on the surface of the slender body whose local radius is a. The integrated induced velocity at such points must then be equated with the known or assumed translational velocity of the section under consideration. The result will, in general, be a system of complicated integral equations for the strength of the singularity distributions. The first simplification of slender-body theory results from the observation that the velocities induced at A by singularities outside a certain near-field will be dominated by the stokeslets in the far-field since their far-field effect (like r^{-1}) dominates that of the other, singularities. Thus the primary distribution is one of stokeslets along the entire axis of the body. The boundary condition at the cross-section under consideration is satisfied by introducing a potential doublet (or if necessary other singularities) only within the near-field. In particular the integrated effect of singularities with a far-field decay faster than r^{-1} can be fairly accurately determined by terminating the integration at some distance $s = \pm\lambda$ from the section under consideration where $s_1, s_2 \ll \lambda \ll a$, s_1 and s_2 being the distances to the ends of the slender body. On the other hand, the integration for the velocity induced by the stokeslets cannot be truncated in this way and indeed yields a velocity with terms like $\ln(s_1 s_2/a^2)$. The reader is referred to Lighthill (1975, p.49) for the forms of the integrated induced velocities. Note that this is another manifestation of Stokes paradox for the translation of an infinitely long cylinder; when s_1 or s_2 tend to infinity, the boundary condition at the section under consideration cannot be satisfied. We must also note that such a construction is limited to sections sufficiently far from the ends of the slender body; Tillett (1970) has examined some of the problems associated with such "end effects."

The net result of these considerations is that one must seek the strength and direction of stokeslets distributed along the entire axis of the slender body plus the local distribution of higher-order singularities that satisfies the required boundary condition at every point on the slender-body surface. A useful approximate way of implementing this was suggested by Lighthill (1975) and by Johnson and Wu (private communication). If the local radius of curvature of the body is large compared with a, then the combined effects of both the near- and far-field distributions may be replaced by a distribution of stokeslets alone in the far-field regions, $s > \delta$ and $s < \delta$. For the components of the stokeslets normal to the axis $\delta = a/2e^{1/2}$ whereas for the components tangential to the axis $\delta = ae^{1/2}/2$. This observation considerably simplifies the algebra required in obtaining solutions for the motions of slender bodies of more complicated geometry.

The simplest solutions are those for the translation of straight slender cylinders as obtained by Tillett (1970) and Cox (1970). Defining force coefficients as the force per unit length of the body divided by the translational velocity, U, Cox (1970) improved on the original work of Burgers (1938) and Broersma (1960) to show that the force coefficient for a cylinder, with length 2ℓ and maximum radius a, moving perpendicular to its axis was

$$C_n = \frac{4\pi\mu}{\ln 2\ell/a + C_1} + O\left[\frac{\mu}{(\ln \ell/a)^3}\right] \tag{Bld9}$$

while that for motion parallel with its axis, C_s, was

$$C_s = \frac{2\pi\mu}{\ln 2\ell/a + C_2} + O\left[\frac{\mu}{(\ln \ell/a)^3}\right] \tag{Bld10}$$

The value of C_2 was $(C_1 - 1)$ and the value of C_1 depended on the axial variation of the radius of the cylinder. A uniform axial cylinder took a value $C_1 = \ln 2 - (1/2) = 0.193$, whereas a prolate spheroid yielded $C_1 = 1/2$. The latter agrees with the results of the exact solution for a spheroid, equations (Blc7) and (Blc8); in this case the answers are more accurate than the error terms in equations (Bld9) and (Bld10) indicate.