Compressible Potential Flow

Compressible potential flow refers to a methodology used to solve steady, irrotational, compressible flows in which the direction of the velocity vector is always close to the coordinate direction, x_1, of the oncoming uniform stream of velocity, U. As we shall see it includes a range of both subsonic and supersonic flows but excludes a range of transonic flows around $M = 1$. It allows existing solutions for steady, incompressible planar and three-dimensional flows to be adapted for use as solutions for these ranges of steady, compressible flow. Then the basic conservation equations become

- **Continuity:**

$$\frac{\partial(\rho u_i)}{\partial x_i} \;=\; 0 \qquad \text{or} \qquad u_i \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial u_i}{\partial x_i} \;=\; 0 \tag{Bon1}$$

- **Momentum:** In the absence of gravity, the unsteady terms and the viscous terms, the momentum equation becomes

$$\rho u_j \frac{\partial u_i}{\partial x_j} \;=\; -\frac{\partial p}{\partial x_i} \;=\; -\frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial x_i} \tag{Bon2}$$

  where the second version requires specification of the thermodynamic derivative, $\partial p/\partial \rho$. Since these flows usually involve small changes, it is appropriate to assume that this should be the isentropic derivative so that, denoting the isentropic speed of sound by c, equation (Bon2) becomes

$$u_j \frac{\partial u_i}{\partial x_j} \;=\; -\frac{c^2}{\rho}\frac{\partial \rho}{\partial x_i} \;=\; c^2 \frac{\partial u_i}{\partial x_i} \tag{Bon3}$$

  where the second version follows by using the continuity equation (Bon1).

- **Irrotationality:** If the flow is irrotational, this allows the definition of a velocity potential (see section (Bga)), ϕ, such that

$$u_i = \frac{\partial \phi}{\partial x_i} \tag{Bon4}$$

  so that the governing equation (Bon3) may be written as

$$\frac{\partial \phi}{\partial x_j}\frac{\partial^2 \phi}{\partial x_i \partial x_j} \;=\; c^2 \frac{\partial^2 \phi}{\partial x_i \partial x_i} \tag{Bon5}$$

  Note that if the flow is almost incompressible so the $c^2 \gg u_i u_j$ then equation (Bon5) reduces to the equation for incompressible flow, $\nabla^2 \phi = 0$.

The next step is to apply the small perturbation assumption described in the introduction. The velocity components in the $i = 1$, 2 and 3 directions are denoted by u_1, u_2 and u_3 where

$$u_1 \;\approx\; U \qquad \text{and} \qquad u_1 - U \ll U \quad ; \quad u_2 \ll U \quad ; \quad u_3 \ll U \tag{Bon6}$$

and U is the constant free stream velocity. Then, though equation (Bon5) comprises nine different equations, only one is dominant namely

$$U^2 \frac{\partial^2 \phi}{\partial x_1^2} \;=\; c^2 \left\{ \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2} \right\} \tag{Bon7}$$

or

$$\frac{\partial^2\phi}{\partial x_1^2}+\frac{\partial^2\phi}{\partial x_2^2}+\frac{\partial^2\phi}{\partial x_3^2} \;=\; \frac{U^2}{c^2}\frac{\partial^2\phi}{\partial x_1^2} \tag{Bon8}$$

Even though the c^2 used in this equation was earlier defined as a local property that may therefore vary from point to point within the flow, it is consistent with the level of approximation in this analysis to approximate U^2/c^2 by M^2 where M is the Mach number of the upstream flow. Thus the governing equation becomes

$$(1-M^2)\frac{\partial^2\phi}{\partial x_1^2}+\frac{\partial^2\phi}{\partial x_2^2}+\frac{\partial^2\phi}{\partial x_3^2} \;=\; 0 \tag{Bon9}$$

and this is the fundamental governing equation of compressible potential flow. We note immediately that when $M<1$ this is an elliptic partial differential equation for the flow whereas, when $M>1$ this is a hyperbolic partial differential equation whose solutions are different in type from those of an elliptic equation. However, there will also be a region close to $M=1$ where the nature of the flow is less clear, where the approximations used in the above derivation may become less appropriate and where solutions may be difficult to obtain.

The next step is to establish boundary condition for these steady, irrotational, compressible flows and for simplicity we will confine to attention to planar flows. The notation is depicted in Figure 1. Then,

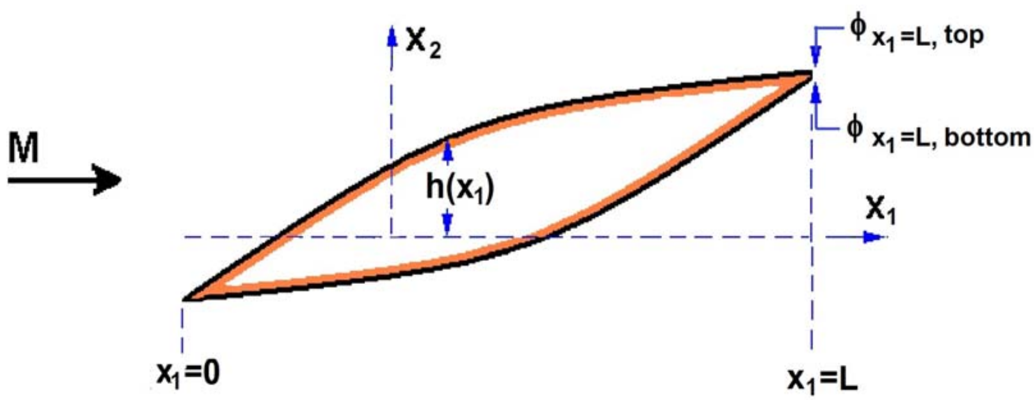

Figure 1: Small perturbation flow notation.

consistent with the level of approximation in this analysis, the condition that the flow at the solid surface is tangential to the surface becomes

$$\left(\frac{\partial\phi}{\partial x_2}\right)_{x_2=h} \;=\; (u_2)_{x_2=h} \;=\; U\left(\frac{\partial h}{\partial x_1}\right) \tag{Bon10}$$

which, again consistent with the level of approximation in this analysis, is often approximated by

$$\left(\frac{\partial\phi}{\partial x_2}\right)_{x_2=0} \;=\; U\left(\frac{\partial h}{\partial x_1}\right) \tag{Bon11}$$

where the right-hand side is a known input.

Once the solution, $\phi(x_1,x_2)$, has been obtained, the pressures acting at the solid surface are often desired. The coefficient of pressure, C_p, is given by

$$C_p \;=\; \frac{p-p_\infty}{\frac{1}{2}\rho_\infty U^2} \;=\; \frac{2}{\gamma M^2}\left\{\frac{p-p_\infty}{p_\infty}\right\} \tag{Bon12}$$

where p_∞ and ρ_∞ (and T_∞) are the flow conditions far upstream (where $u_1 = U$) and the second version follows from $c^2 = \gamma p_\infty/\rho_\infty$. Moreover, provided $p - p_\infty$ and $T - T_\infty$ are small it follows from the isentropic relations and from the energy equation that

$$\frac{p - p_\infty}{p_\infty} = \frac{\gamma}{\gamma - 1}\frac{T - T_\infty}{T_\infty} \qquad \text{and} \qquad \frac{T - T_\infty}{T_\infty} = -\frac{U(u_1 - U)}{c_p} \tag{Bon13}$$

so that from equations (Bon12) and (Bon13)

$$C_p = -\frac{2(u_1 - U)}{U} = -\frac{2}{U}\left\{\left(\frac{\partial \phi}{\partial x_1}\right)_{x_2=h} - U\right\} \tag{Bon14}$$

and therefore the pressure coefficients (on both the upper and lower surfaces) can be calculated once the solution, $\phi(x_1, x_2)$, has been obtained.

Moreover, for a slender body, extending from $x_1 = 0$ to $x_1 = L$, the lift coefficient will then follow as

$$C_L = \frac{1}{L}\int_0^L \left\{(C_p)_L - (C_p)_U\right\} \, dx_1 \tag{Bon15}$$

where the subscripts U and L denote the upper and lower surfaces respectively. Substituting for C_p and integrating this yields

$$C_L = \frac{2}{LU}\left\{\phi_{x_1=L, x_2=h_L, L} - \phi_{x_1=L, x_2=h_U, U}\right\} \tag{Bon16}$$

where $x_2 = h_L(x_1)$ defines the lower surface and $x_2 = h_U(x_1)$ defines the upper surface. Equation (Bon16) assumes that ϕ at the leading edge ($x_1 = 0$) is the same for both the upper and lower surfaces.

The current section has identified the basic equations, boundary conditions and result computations for compressible potential flow. However, the specifics for subsonic and supersonic solutions differ because of the different nature of the two types of flows. In the two sections that follow we address some of those specifics, starting in the next section (Boo) with the hyperbolic equations of supersonic flow and ending in section (Bop) with the elliptic equations of subsonic flow.